数値計算・講義資料—数列・級数の加速—

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Richardson 加速　数値微分

$$f'(x) \simeq a_n \equiv \frac{f(x+2^{-n}h)-f(x)}{2^{-n}h}$$

に Richardson 加速

$$a_n^{(0)} = a_n, \quad a_n^{(m)} = \frac{a_n^{(m-1)} - 2^{-m}a_{n-1}^{(m-1)}}{1-2^{-m}}, \quad m=1,2,\ldots;\ n=m,m+1,m+2,\ldots$$

を適用した結果を下記に記す (`a(m,n)`$= a_n^{(m)}$)．ただし，

$$f(x) = \mathrm{e}^x, \quad x=0, \quad h=0.25$$

である．Richardson 加速により収束が速くなっていることが確認される．

```
                                           relative error
----------------------------------------------------------
a(   0,    0) = 1.136101666750966e+000  1.361e-001
a(   0,    1) = 1.065187624534611e+000  6.519e-002
a(   0,    2) = 1.031911342685749e+000  3.191e-002
a(   0,    3) = 1.015789039971288e+000  1.579e-002
a(   0,    4) = 1.007853349547887e+000  7.853e-003
a(   0,    5) = 1.003916442425350e+000  3.916e-003
a(   0,    6) = 1.001955670616951e+000  1.956e-003
a(   0,    7) = 1.000977198593432e+000  9.772e-004

a(   1,    1) = 9.942735823182556e-001 -5.726e-003
a(   1,    2) = 9.986350608368877e-001 -1.365e-003
a(   1,    3) = 9.996667372568275e-001 -3.333e-004
a(   1,    4) = 9.999176591244847e-001 -8.234e-005
a(   1,    5) = 9.999795353028134e-001 -2.046e-005
a(   1,    6) = 9.999948988085521e-001 -5.101e-006
a(   1,    7) = 9.999987265699133e-001 -1.273e-006

a(   2,    2) = 1.000088887009765e+000  8.889e-005
a(   2,    3) = 1.000010629396807e+000  1.063e-005
a(   2,    4) = 1.000001299747037e+000  1.300e-006
a(   2,    5) = 1.000000160695590e+000  1.607e-007
a(   2,    6) = 1.000000019977132e+000  1.998e-008
a(   2,    7) = 1.000000002490367e+000  2.490e-009

a(   3,    3) = 9.999994497378134e-001 -5.503e-007
```

```
a(  3,   4) = 9.999999669399271e-001 -3.306e-008
a(  3,   5) = 9.999999979739541e-001 -2.026e-009
a(  3,   6) = 9.999999998744950e-001 -1.255e-010
a(  3,   7) = 9.999999999922578e-001 -7.742e-012

a(  4,   4) = 1.000000001420068e+000  1.420e-009
a(  4,   5) = 1.000000000042889e+000  4.289e-011
a(  4,   6) = 1.000000000001198e+000  1.198e-012
a(  4,   7) = 1.000000000000109e+000  1.086e-013

a(  5,   5) = 9.999999999984641e-001 -1.536e-012
a(  5,   6) = 9.999999999998528e-001 -1.472e-013
a(  5,   7) = 1.000000000000074e+000  7.350e-014

a(  6,   6) = 9.999999999998749e-001 -1.251e-013
a(  6,   7) = 1.000000000000077e+000  7.705e-014

a(  7,   7) = 1.000000000000079e+000  7.860e-014
```

Romberg 積分

$$\int_0^1 \frac{\mathrm{d}x}{1+x^2} \simeq S_n \equiv h\left\{\frac{1}{2}f(0) + \sum_{k=1}^{2^n-1} f(kh) + \frac{1}{2}f(1)\right\}, \quad h = 2^{-n}, \quad n = 1, 2, \ldots$$

に Romberg 積分

$$S_n^{(0)} = S_n, \quad S_n^{(m)} = \frac{S_n^{(m-1)} - 4^{-m} S_{n-1}^{(m-1)}}{1 - 4^{-m}}, \quad m = 1, 2, \ldots; \, n = m, m+1, m+2, \ldots$$

を適用した結果を下記に記す (S(m,n)$= S_n^{(m)}$). Romberg 積分により収束が速くなっていることが確認される.

```
S(exact) = 7.853981633974483e-001
                                          relative error
-------------------------------------------------------
S(  0,   0) = 7.827941176470589e-001 -3.316e-003
S(  0,   1) = 7.847471236227722e-001 -8.289e-004
S(  0,   2) = 7.852354030103472e-001 -2.072e-004
S(  0,   3) = 7.853574732937438e-001 -5.181e-005
S(  0,   4) = 7.853879908714134e-001 -1.295e-005
S(  0,   5) = 7.853956202659381e-001 -3.238e-006

S(  1,   1) = 7.853981256146767e-001 -4.811e-008
S(  1,   2) = 7.853981628062056e-001 -7.528e-010
S(  1,   3) = 7.853981633882093e-001 -1.176e-011
S(  1,   4) = 7.853981633973033e-001 -1.846e-013
```

```
S(  1,   5) = 7.853981633974464e-001 -2.403e-015

S(  2,   2) = 7.853981652856408e-001  2.404e-009
S(  2,   3) = 7.853981634270096e-001  3.764e-011
S(  2,   4) = 7.853981633979096e-001  5.873e-013
S(  2,   5) = 7.853981633974559e-001  9.754e-015

S(  3,   3) = 7.853981633975076e-001  7.549e-014
S(  3,   4) = 7.853981633974477e-001 -7.068e-016
S(  3,   5) = 7.853981633974487e-001  5.654e-016

S(  4,   4) = 7.853981633974475e-001 -9.895e-016
S(  4,   5) = 7.853981633974487e-001  5.654e-016

S(  5,   5) = 7.853981633974487e-001  5.654e-016
```

Euler 変換

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2.$$

の部分和

$$S_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}, \quad n = 0, 1, 2, \ldots$$

を計算した結果を下記に記す．なかなか収束しない．

```
S(exact) = 6.931471805599453e-001
                                        relative error
---------------------------------------------
S(10) = 7.365440115440117e-001  6.261e-002
S(15) = 6.628718503718505e-001 -4.368e-002
S(20) = 7.163904507944757e-001  3.353e-002
S(25) = 6.742859610812905e-001 -2.721e-002
S(30) = 7.090162022075272e-001  2.289e-002
S(35) = 6.794511185983259e-001 -1.976e-002
S(40) = 7.051936256951333e-001  1.738e-002
```

$\{S_n\}$ に Euler 変換

$$S_n^{(0)} = S_n, \quad S_n^{(m)} = \frac{S_n^{(m-1)} + S_{n-1}^{(m-1)}}{2}, \quad m = 1, 2, \ldots; \, n = m, m+1, m+2, \ldots$$

を施した結果を下記に記す (`S( m, n)`$= S_n^{(m)}$)．収束が速くなっていることが確認される．

```
                                             relative error
---------------------------------------------------
S(  0,   0) = 1.000000000000000e+000 4.427e-001
```

```
S(  1,   1) = 7.500000000000000e-001 8.202e-002
S(  2,   2) = 7.083333333333334e-001 2.191e-002
S(  3,   3) = 6.979166666666667e-001 6.881e-003
S(  4,   4) = 6.947916666666667e-001 2.372e-003
S(  5,   5) = 6.937500000000000e-001 8.697e-004
S( 10,  10) = 6.931536345598846e-001 9.311e-006
S( 15,  15) = 6.931472819393201e-001 1.463e-007
S( 20,  20) = 6.931471824640917e-001 2.747e-009
S( 25,  25) = 6.931471805996439e-001 5.727e-011
S( 30,  30) = 6.931471805608318e-001 1.279e-012
S( 35,  35) = 6.931471805599662e-001 3.011e-014
S( 40,  40) = 6.931471805599460e-001 9.610e-016
```

Aitken 加速　数値積分（台形則）

$$\int_0^1 \frac{\mathrm{d}x}{1+x^2} \simeq S_n \equiv h\left\{\frac{1}{2}f(0)+\sum_{k=1}^{2^n-1} f(kh)+\frac{1}{2}f(1)\right\}, \quad h=2^{-n}, \quad n=1,2,\ldots$$

に Aitken 加速

$$S_n^{(0)}=S_n, \quad S_n^{(m)}=S_n^{(m-1)}-\frac{(S_n^{(m-1)}-S_{n-1}^{(m-1)})^2}{S_n^{(m-1)}-2S_{n-1}^{(m-1)}+S_{n-2}^{(m-1)}}, \quad m=1,2,\ldots;n=2m,2m{+}1,2m{+}2,\ldots$$

を施した結果を下記に記す (`S(m,n)`$=S_n^{(m)}$)．Aitken 加速により収束が速くなっていることが確認される．

```
S(exact) = 7.853981633974483e-001
                                            relative error
----------------------------------------------------
S(  0,   0) = 7.827941176470589e-001 -3.316e-003
S(  0,   1) = 7.847471236227722e-001 -8.289e-004
S(  0,   2) = 7.852354030103472e-001 -2.072e-004
S(  0,   3) = 7.853574732937438e-001 -5.181e-005
S(  0,   4) = 7.853879908714134e-001 -1.295e-005
S(  0,   5) = 7.853956202659381e-001 -3.238e-006

S(  1,   2) = 7.853981752043262e-001 1.503e-008
S(  1,   3) = 7.853981635822116e-001 2.352e-010
S(  1,   4) = 7.853981634003346e-001 3.675e-012
S(  1,   5) = 7.853981633974940e-001 5.824e-014

S(  2,   4) = 7.853981633974432e-001 -6.502e-015
S(  2,   5) = 7.853981633974490e-001 8.481e-016
```