

数値計算・講義資料—数列・級数の加速—

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Richardson 加速 数値微分

$$f'(x) \simeq a_n \equiv \frac{f(x+2^{-n}h) - f(x)}{2^{-n}h}$$

に Richardson 加速

$$a_n^{(0)} = a_n, \quad a_n^{(m)} = \frac{a_n^{(m-1)} - 2^{-m}a_{n-1}^{(m-1)}}{1 - 2^{-m}}, \quad m = 1, 2, \dots; n = m, m+1, m+2, \dots$$

を適用した結果を下記に記す ($\mathbf{a}(m, n) = a_n^{(m)}$). ただし,

$$f(x) = e^x, \quad x = 0, \quad h = 0.25$$

である. Richardson 加速により収束が速くなっていることが確認される.

relative error

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a( 0,  0) = 1.136101666750966e+000  1.361e-001  
a( 0,  1) = 1.065187624534611e+000  6.519e-002  
a( 0,  2) = 1.031911342685749e+000  3.191e-002  
a( 0,  3) = 1.015789039971288e+000  1.579e-002  
a( 0,  4) = 1.007853349547887e+000  7.853e-003  
a( 0,  5) = 1.003916442425350e+000  3.916e-003  
a( 0,  6) = 1.001955670616951e+000  1.956e-003  
a( 0,  7) = 1.000977198593432e+000  9.772e-004  
  
a( 1,  1) = 9.942735823182556e-001 -5.726e-003  
a( 1,  2) = 9.986350608368877e-001 -1.365e-003  
a( 1,  3) = 9.996667372568275e-001 -3.333e-004  
a( 1,  4) = 9.999176591244847e-001 -8.234e-005  
a( 1,  5) = 9.999795353028134e-001 -2.046e-005  
a( 1,  6) = 9.999948988085521e-001 -5.101e-006  
a( 1,  7) = 9.999987265699133e-001 -1.273e-006  
  
a( 2,  2) = 1.000088887009765e+000  8.889e-005  
a( 2,  3) = 1.000010629396807e+000  1.063e-005  
a( 2,  4) = 1.000001299747037e+000  1.300e-006  
a( 2,  5) = 1.000000160695590e+000  1.607e-007  
a( 2,  6) = 1.000000019977132e+000  1.998e-008  
a( 2,  7) = 1.000000002490367e+000  2.490e-009  
  
a( 3,  3) = 9.999994497378134e-001 -5.503e-007
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a(3, 4) = 9.999999669399271e-001 -3.306e-008
a(3, 5) = 9.99999979739541e-001 -2.026e-009
a(3, 6) = 9.99999998744950e-001 -1.255e-010
a(3, 7) = 9.9999999922578e-001 -7.742e-012

a(4, 4) = 1.000000001420068e+000 1.420e-009
a(4, 5) = 1.000000000042889e+000 4.289e-011
a(4, 6) = 1.000000000001198e+000 1.198e-012
a(4, 7) = 1.000000000000109e+000 1.086e-013

a(5, 5) = 9.99999999984641e-001 -1.536e-012
a(5, 6) = 9.9999999998528e-001 -1.472e-013
a(5, 7) = 1.000000000000074e+000 7.350e-014

a(6, 6) = 9.9999999998749e-001 -1.251e-013
a(6, 7) = 1.000000000000077e+000 7.705e-014

a(7, 7) = 1.000000000000079e+000 7.860e-014

Romberg 積分

$$\int_0^1 \frac{dx}{1+x^2} \simeq S_n \equiv h \left\{ \frac{1}{2}f(0) + \sum_{k=1}^{2^n-1} f(kh) + \frac{1}{2}f(1) \right\}, \quad h = 2^{-n}, \quad n = 1, 2, \dots$$

に Romberg 積分

$$S_n^{(0)} = S_n, \quad S_n^{(m)} = \frac{S_n^{(m-1)} - 4^{-m}S_{n-1}^{(m-1)}}{1 - 4^{-m}}, \quad m = 1, 2, \dots; \quad n = m, m+1, m+2, \dots$$

を適用した結果を下記に記す ($S(m, n) = S_n^{(m)}$). Romberg 積分により収束が速くなっていることが確認される.

S(exact) = 7.853981633974483e-001

relative error

S(0, 0) = 7.827941176470589e-001 -3.316e-003
S(0, 1) = 7.847471236227722e-001 -8.289e-004
S(0, 2) = 7.852354030103472e-001 -2.072e-004
S(0, 3) = 7.853574732937438e-001 -5.181e-005
S(0, 4) = 7.853879908714134e-001 -1.295e-005
S(0, 5) = 7.853956202659381e-001 -3.238e-006

S(1, 1) = 7.853981256146767e-001 -4.811e-008
S(1, 2) = 7.853981628062056e-001 -7.528e-010
S(1, 3) = 7.853981633882093e-001 -1.176e-011
S(1, 4) = 7.853981633973033e-001 -1.846e-013

S(1, 5) = 7.853981633974464e-001 -2.403e-015

S(2, 2) = 7.853981652856408e-001 2.404e-009

S(2, 3) = 7.853981634270096e-001 3.764e-011

S(2, 4) = 7.853981633979096e-001 5.873e-013

S(2, 5) = 7.853981633974559e-001 9.754e-015

S(3, 3) = 7.853981633975076e-001 7.549e-014

S(3, 4) = 7.853981633974477e-001 -7.068e-016

S(3, 5) = 7.853981633974487e-001 5.654e-016

S(4, 4) = 7.853981633974475e-001 -9.895e-016

S(4, 5) = 7.853981633974487e-001 5.654e-016

S(5, 5) = 7.853981633974487e-001 5.654e-016

Euler 変換

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2.$$

の部分

$$S_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k}, \quad n = 0, 1, 2, \dots$$

を計算した結果を下記に記す. なかなか収束しない.

S(exact) = 6.931471805599453e-001

relative error

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S(10) = 7.365440115440117e-001 6.261e-002
S(15) = 6.628718503718505e-001 -4.368e-002
S(20) = 7.163904507944757e-001 3.353e-002
S(25) = 6.742859610812905e-001 -2.721e-002
S(30) = 7.090162022075272e-001 2.289e-002
S(35) = 6.794511185983259e-001 -1.976e-002
S(40) = 7.051936256951333e-001 1.738e-002

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{S_n} に Euler 変換

$$S_n^{(0)} = S_n, \quad S_n^{(m)} = \frac{S_n^{(m-1)} + S_{n-1}^{(m-1)}}{2}, \quad m = 1, 2, \dots; n = m, m+1, m+2, \dots$$

を施した結果を下記に記す (S(m, n)=S_n^(m)). 収束が速くなっていることが確認される.

relative error

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S( 0, 0) = 1.000000000000000e+000 4.427e-001

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S(1, 1) = 7.500000000000000e-001 8.202e-002
 S(2, 2) = 7.083333333333334e-001 2.191e-002
 S(3, 3) = 6.979166666666667e-001 6.881e-003
 S(4, 4) = 6.947916666666667e-001 2.372e-003
 S(5, 5) = 6.937500000000000e-001 8.697e-004
 S(10, 10) = 6.931536345598846e-001 9.311e-006
 S(15, 15) = 6.931472819393201e-001 1.463e-007
 S(20, 20) = 6.931471824640917e-001 2.747e-009
 S(25, 25) = 6.931471805996439e-001 5.727e-011
 S(30, 30) = 6.931471805608318e-001 1.279e-012
 S(35, 35) = 6.931471805599662e-001 3.011e-014
 S(40, 40) = 6.931471805599460e-001 9.610e-016

Aitken 加速 数值積分 (台形則)

$$\int_0^1 \frac{dx}{1+x^2} \simeq S_n \equiv h \left\{ \frac{1}{2}f(0) + \sum_{k=1}^{2^n-1} f(kh) + \frac{1}{2}f(1) \right\}, \quad h = 2^{-n}, \quad n = 1, 2, \dots$$

に Aitken 加速

$$S_n^{(0)} = S_n, \quad S_n^{(m)} = S_n^{(m-1)} - \frac{(S_n^{(m-1)} - S_{n-1}^{(m-1)})^2}{S_n^{(m-1)} - 2S_{n-1}^{(m-1)} + S_{n-2}^{(m-1)}}, \quad m = 1, 2, \dots; n = 2m, 2m+1, 2m+2, \dots$$

を施した結果を下記に記す ($S(m, n) = S_n^{(m)}$). Aitken 加速により収束が速くなっていることが確認される.

S(exact) = 7.853981633974483e-001

relative error

S(0, 0) = 7.827941176470589e-001 -3.316e-003
 S(0, 1) = 7.847471236227722e-001 -8.289e-004
 S(0, 2) = 7.852354030103472e-001 -2.072e-004
 S(0, 3) = 7.853574732937438e-001 -5.181e-005
 S(0, 4) = 7.853879908714134e-001 -1.295e-005
 S(0, 5) = 7.853956202659381e-001 -3.238e-006

S(1, 2) = 7.853981752043262e-001 1.503e-008
 S(1, 3) = 7.853981635822116e-001 2.352e-010
 S(1, 4) = 7.853981634003346e-001 3.675e-012
 S(1, 5) = 7.853981633974940e-001 5.824e-014

S(2, 4) = 7.853981633974432e-001 -6.502e-015
 S(2, 5) = 7.853981633974490e-001 8.481e-016