解析学・宿題・解答

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第 1 問　次の不定積分を求めよ（(1), (2) は答だけ記せばよい）．

$$(1)\quad \int e^{2x}dx,\quad (2)\quad \int \sin 3x\, dx,\quad (3)\quad \int x\cos 2x\, dx,\quad (4)\quad \int e^x \sin x\, dx,$$

$$(5)\quad \int \frac{dx}{1+x^2},\quad (6)\quad \int \frac{dx}{\sqrt{a^2-x^2}}\quad (a \text{ は正の定数}),$$

$$(7)\quad \int \frac{dx}{\sqrt{1+x^2}},\quad (8)\quad \int \sqrt{1+x^2}dx.$$

以下，C は積分定数とする．

(1). $\dfrac{1}{2}e^{2x}+C.$

(2). $-\dfrac{1}{x}\cos 3x+C.$

(3). 部分積分を用いる．

$$\begin{aligned}\int x\cos 2x\, dx &= \frac{1}{2}\int x(\sin 2x)'\, dx = \frac{1}{2}\left\{x\sin 2x - \int \sin 2x\, dx\right\}\\ &= \frac{1}{2}\left\{x\sin 2x + \frac{1}{2}\cos 2x + C\right\} = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.\end{aligned}$$

(4). $\displaystyle\int e^x\cos x\, dx$ とセットで解く．

$$\int e^x \sin x\, dx = -\int e^x(\cos x)'\, dx = -\left\{e^x\cos x - \int e^x \cos x\, dx\right\}$$

により，

$$\int e^x\sin x\, dx - \int e^x\cos x\, dx = -e^x\cos x. \tag{1}$$

$$\int e^x\cos x\, dx = \int e^x(\sin x)'\, dx = e^x\sin x - \int e^x \sin x\, dx$$

により，

$$\int e^x\sin x\, dx + \int e^x\cos x\, dx = e^x\sin x. \tag{2}$$

(1), (2) より $\displaystyle\int e^x\cos x\, dx$ を消去して，

$$\int e^x\sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

(5). 置換積分により解く．$x=\tan\theta$ と置くと，$dx = \dfrac{d\theta}{\cos^2\theta}$ により

$$\int\frac{dx}{1+x^2} = \int\frac{1}{1+\tan^2\theta}\frac{d\theta}{\cos^2\theta} = \int\cos^2\theta\frac{d\theta}{\cos^2\theta} = \int d\theta = \theta + C.$$

$$\therefore\quad \int\frac{dx}{1+x^2} = \arctan x + C.$$

(6). 置換積分により解く．$x = a\sin\theta$ と置くと，$\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$ により

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\,\mathrm{d}\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta\,\mathrm{d}\theta}{a\cos\theta} = \int \mathrm{d}\theta = \theta + C.$$

$$\therefore \quad \int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C.$$

(7). 置換積分により解く．$x = \sinh u$ と置くと，$\mathrm{d}x = \cosh u\,\mathrm{d}u$ により

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \int \frac{\sinh u\mathrm{d}u}{1+\sinh^2 u} = \int \frac{\sinh u\mathrm{d}u}{\cosh u} = \int \mathrm{d}u = u + C.$$

u を x で表す.

$$\sinh u = \frac{\mathrm{e}^u - \mathrm{e}^{-u}}{2} = x, \quad (\mathrm{e}^u)^2 - 2x\mathrm{e}^u - 1 = 0, \quad \mathrm{e}^u = x \pm \sqrt{x^2+1},$$

$\mathrm{e}^u > 0$ より $\mathrm{e}^u = x + \sqrt{x^2+1}$，$u = \log(x + \sqrt{x^2}+1)$.

$$\therefore \quad \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \log(x + \sqrt{x^2+1}) + C.$$

(8). 前問の結果を用いる.

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \int (x)'\sqrt{1+x^2}\,\mathrm{d}x = x\sqrt{1+x^2} - \int x\cdot\frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$= x\sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}x = x\sqrt{1+x^2} - \int \frac{(1+x^2)-1}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = x\sqrt{1+x^2} + \log(x+\sqrt{1+x^2}) - \int \sqrt{1+x^2}\mathrm{d}x.$$

$\int \sqrt{1+x^2}\,\mathrm{dx}$ について解いて，

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\log(x+\sqrt{1+x^2}) + C.$$

第 2 問　次の常微分方程式の一般解を求めよ.

$$(1) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}, \quad (2) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x^2}, \quad (3) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2,$$

$$(4) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2, \quad (5) \quad \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{m\omega^2}{2}x^2 = E \quad (m,\,\omega,\,E \text{ は正の定数}).$$

(1).

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{x}, \quad \int\frac{\mathrm{d}y}{y} = \int\frac{\mathrm{d}x}{x}, \quad \log|y| = \log|x| + c \quad (\,c:\ \text{const.}\,), \quad |y| = \mathrm{e}^c|x|,$$

$$y = Cx \quad (\,C = \pm\mathrm{e}^c \neq 0\,).$$

上式で $C = 0$ と置いた場合の $y \equiv 0$ ももとの方程式を満たす.

$$\therefore \quad y = Cx \quad (C \text{ は任意定数}).$$

(2).

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{x^2}, \quad \int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x^2}, \quad \log|y| = -\frac{1}{x} + c \quad (\, c:\ \text{const.}\,),$$
$$y = C\mathrm{e}^{-1/x} \quad (\, C = \pm\mathrm{e}^{c} \neq 0\,).$$

上式で $C = 0$ と置いた場合の $y \equiv 0$ ももとの方程式を満たす.

$$\therefore \quad y = C\mathrm{e}^{-1/x} \quad (C \text{ は任意定数}).$$

(3).

$$\frac{\mathrm{d}y}{1-y^2} = \mathrm{d}x, \quad \int \frac{\mathrm{d}y}{1-y^2} = \int \mathrm{d}x = x + c \quad (\quad \text{c:} \quad \text{const.} \quad),$$
$$\text{左辺} = \frac{1}{2}\int \left(\frac{1}{1-y} + \frac{1}{1+y}\right)\mathrm{d}y = \frac{1}{2}\log\left|\frac{1+y}{1-y}\right|$$

により,

$$\frac{1+y}{1-y} = C\mathrm{e}^{\pm 2x} \quad (\, C = \pm\mathrm{e}^{2c} \neq 0\,), \quad y = \frac{1 - C\mathrm{e}^{\pm 2x}}{1 + C\mathrm{e}^{\pm 2x}}.$$

上式で $C = 0$ と置いた場合の $y \equiv 0$ ももとの方程式を満たす.

$$\therefore \quad y = \frac{C\mathrm{e}^{\pm 2x} - 1}{C\mathrm{e}^{\pm 2x} + 1} \quad (C \text{ は任意定数}).$$

(4).

$$\frac{\mathrm{d}y}{1+y^2} = \mathrm{d}x, \quad \int \frac{\mathrm{d}y}{1+y^2} = \int \mathrm{d}x = x + c \quad (\quad \text{c: const.}\,), \quad \arctan y = x + c.$$
$$\therefore \quad y = \tan(x + c) \quad (c \text{ は任意定数}).$$

(5).

$$\frac{1}{\omega^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = a^2 - x^2 \quad \left(a = \sqrt{\frac{2E}{m\omega^2}}\right), \quad \frac{1}{\omega}\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{a^2 - x^2},$$
$$\frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \pm\mathrm{d}t, \quad \int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \pm\omega \int \mathrm{d}t = \pm\omega(t - t_0) \quad (\, t_0:\ \text{const.}\,),$$
$$\arcsin\frac{x}{a} = \pm\omega(t - t_0).$$
$$\therefore \quad x = \pm a \sin\omega(t - t_0) \quad \left(a = \sqrt{\frac{2E}{m\omega^2}}, \quad t_0 \text{ は任意定数}\right).$$