$$\int\sqrt{\frac{y}{c-y}}\,dy \quad (c>0\ \text{const.})$$

$z=\sqrt{\dfrac{y}{c-y}}$ とおくと, $z^2(c-y)=y$, $y(1+z^2)=cz^2$

$$y=\frac{cz^2}{1+z^2}$$

$$dy=\left\{\frac{2cz}{1+z^2}-\frac{cz^2\cdot 2z}{(1+z^2)^2}\right\}dz=\frac{2cz}{1+z^2}\left(1-\frac{z^2}{1+z^2}\right)dz=\frac{2cz}{(1+z^2)^2}dz.$$

$$\int\sqrt{\frac{y}{c-z}}\,dy=\int\frac{2cz^2}{(1+z^2)^2}dz=2c\int\frac{(1+z^2)-1}{(1+z^2)^2}dz$$

$$=2c\left\{\int\frac{dz}{1+z^2}-\int\frac{dz}{(1+z^2)^2}\right\}.$$

$z=\tan\theta$ とおくと.

$$\int\frac{dz}{1+z^2}=\int\cos^2\theta\frac{d\theta}{\cos^2\theta}=\int d\theta=\theta\ (+\text{Const.}),$$

$$\int\frac{dz}{(1+z^2)^2}=\int\cos^4\theta\frac{d\theta}{\cos^2\theta}=\int\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\int(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)(+\text{const.}),$$

$$\int\sqrt{\frac{y}{c-y}}\,dy=2c\cdot\left(\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right)=\frac{c}{2}(2\theta-\sin 2\theta)\ (+\text{const.}).$$

$$\left(y=\frac{c\tan^2\theta}{1+\tan^2\theta}=c\tan^2\theta\cdot\cos^2\theta=c\sin^2\theta=\frac{c}{2}(1-\cos 2\theta)\right)$$