常微分方程式の数値解法

$$① \begin{cases} \dfrac{dy}{dx} = f(x, y) & (x_I \leqq x \leqq x_F) \\ y(x_I) = y_I \end{cases}$$

成分で書くと、

$$\left.\begin{aligned} y_1' &= f_1(x, y_1, y_2, \cdots, y_m) \\ y_2' &= f_2(x, y_1, y_2, \cdots, y_m) \\ &\cdots \\ y_m' &= f_m(x, y_1, y_2, \cdots, y_m) \end{aligned}\right\} (x_I \leqq x \leqq x_F)$$

$$y_1(x_I) = y_1^{(I)}, \quad y_2(x_I) = y_2^{(I)}, \quad \cdots, \quad y_m(x_I) = y_m^{(I)}$$

2階線形常微分方程式の初期値問題

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad (x_I \leqq x \leqq x_F),$$

$$y(x_I) = y_I, \quad y'(x_I) = y_I^{(1)}$$

$$y_1 = y(x), \quad y_2 = y'(x)$$

とおけば

$$\begin{cases} \dfrac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -a_2(x)y_1 - a_1(x)y_2 \end{bmatrix} & (x_F \leqq x \leqq x_I), \\ \begin{bmatrix} y_1(x_I) \\ y_2(x_I) \end{bmatrix} = \begin{bmatrix} y_I \\ y_I^{(1)} \end{bmatrix} \end{cases}$$

…… ① の形に帰着する。

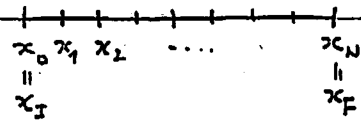
① は次の積分方程式と同等である：

$$y(x) = y_I + \int_{x_I}^{x} f(\xi, y(\xi))\, d\xi \quad (x_I \leqq x \leqq x_F). \qquad ②$$

② の右辺の積分を近似することにより、① の近似解を得る。

区間 $[x_I, x_F]$ を分割する.

$$x_k = x_I + kh \quad \left(k = 0, 1, \dots, N \ ;\ h = \frac{x_F - x_I}{N}\right)$$



$$y_k \simeq y(x_k) \quad (k = 0, 1, \dots, N).$$

① より

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(\xi, y(\xi))\,d\xi \quad (k = 0, 1, \dots, N-1) \ — ②$$

右辺を近似計算することにより, $y_k \simeq y(x_k)$ $(k = 1, 2, \dots, N)$ を順次求めてゆく.

Euler法

・ $\displaystyle\int_{x_k}^{x_{k+1}} f(\xi, y(\xi))\,d\xi \simeq h f(x_k, y(x_k)) \simeq h f(x_k, y_k)$

↓

$$y_{k+1} = y_k + h f(x_k, y_{k+1}) \quad \text{--- 前進Euler法}$$

・ $\displaystyle\int_{x_k}^{x_{k+1}} f(\xi, y(\xi))\,d\xi \simeq h f(x_{k+1}, y(x_{k+1})) \simeq h f(x_{k+1}, y_{k+1})$

↓

$$y_{k+1} = y_k + h f(x_{k+1}, y_{k+1}) \quad \text{--- 後退Euler法}$$

各段で y_{k+1} に関する方程式を解かなければならない.
一方, 数値的安定性は前進Euler法より後退Euler法の方が良い.

Heun法

$$\int_{x_k}^{x_{k+1}} f(\xi, y(\xi))\,d\xi \simeq \frac{h}{2}\left\{f(x_k, y(x_k)) + f(x_{k+1}, y(x_{k+1}))\right\} \quad (\text{台形則})$$

$$y(x_k) \simeq y_k, \quad y(x_{k+1}) \simeq y(x_k) + h y'(x_k) \simeq y_k + h f(x_k, y_k)$$

↓

No.

$k = 0, 1, 2, \cdots, N-1$ に対し

$$
\left[
\begin{aligned}
k_1 &= f(x_k, y_k); \\
k_2 &= f(x_k, y_k + h k_1); \\
y_{k+1} &= y_k + \frac{h}{2}(k_1 + k_2);
\end{aligned}
\right.
$$

Runge-Kutta 法

$$\int_{x_k}^{x_{k+1}} f(\xi, y(\xi))\, d\xi \simeq \frac{h}{6}\left\{ f(x_k, y(x_k)) + 4 f(x_{k+1/2}, y(x_{k+1/2})) + f(x_{k+1}, y(x_{k+1})) \right\}$$

(Simpson 公式, $x_{k+1/2} = x_k + \frac{h}{2}$)

$$y(x_k) \simeq y_k,$$

$$y'(x_k) = f(x_k, y(x_k)) \simeq f(x_k, y_k) \equiv k_1,$$

$$y(x_{k+1/2}) \simeq y(x_k) + \frac{h}{2} y'(x_k) \simeq y_k + \frac{h}{2} k_1,$$

$$y'(x_{k+1/2}) = f(x_{k+1/2}, y(x_{k+1/2})) \simeq f(x_{k+1/2}, y_k + \frac{h}{2} k_1) \equiv k_2,$$

$$y(x_{k+1/2}) \simeq y(x_k) + \frac{h}{2} y'(x_{k+1/2}) \simeq y_k + \frac{h}{2} k_2,$$

$$y'(x_{k+1/2}) = f(x_{k+1/2}, y(x_{k+1/2})) \simeq f(x_{k+1/2}, y_k + \frac{h}{2} k_2) \equiv k_3,$$

$$y(x_{k+1}) \simeq y(x_k) + h y'(x_{k+1/2}) \simeq y_k + h k_3,$$

$$y'(x_{k+1}) = f(x_{k+1}, y(x_{k+1})) \simeq f(x_{k+1}, y_k + h k_3) \equiv k_4,$$

$$\Downarrow$$

$k = 0, 1, 2, \cdots, N-1$ に対し

$$
\left[
\begin{aligned}
k_1 &= f(x_k, y_k); \\
k_2 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2} k_1); \\
k_3 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2} k_2); \\
k_4 &= f(x_k + h, y_k + h k_3); \\
y_{k+1} &= y_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4);
\end{aligned}
\right.
$$

誤差

$$\epsilon(h) \equiv \max_{0 \leq k \leq N} \| y(x_k) - y_k \|$$

$$= \begin{cases} O(h) & \text{Euler 法} \\ O(h^2) & \text{Heun 法} \\ O(h^4) & \text{Runge-Kutta 法} \end{cases}$$