楕円関数論 (3)
Jacobi の楕円関数（複素関数，二重周期性）

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復習

Jacobi の楕円関数

$$x = \mathrm{sn}\, u = \mathrm{sn}(u; k) \quad \overset{\mathrm{def}}{\Longleftrightarrow} \quad u = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$$\left.\begin{aligned} \mathrm{cn}\, u = \mathrm{cn}(u; k) &= \sqrt{1-\mathrm{sn}^2 u} \\ \mathrm{dn}\, u = \mathrm{dn}(u; k) &= \sqrt{1-k^2\,\mathrm{sn}^2 u} \end{aligned}\right\} \quad (\,|u| \leq K\,),$$

$$k\,(\,0 < k < 1\,)\ :\ \text{母数}, \quad k' = \sqrt{1-k^2}\ :\ \text{補母数},$$

加法定理

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\, u\, \mathrm{cn}\, v\, \mathrm{dn}\, v + \mathrm{sn}\, v\, \mathrm{cn}\, u\, \mathrm{dn}\, u}{1-k^2\,\mathrm{sn}^2 u\, \mathrm{sn}^2 v},$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\, u\, \mathrm{cn}\, v - \mathrm{sn}\, u\, \mathrm{dn}\, u\, \mathrm{sn}\, v\, \mathrm{dn}\, v}{1-k^2\,\mathrm{sn}^2 u\, \mathrm{sn}^2 v},$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\, u\, \mathrm{dn}\, v - k^2\,\mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{sn}\, v\, \mathrm{cn}\, v}{1-k^2\,\mathrm{sn}^2 u\, \mathrm{sn}^2 v}.$$

復習

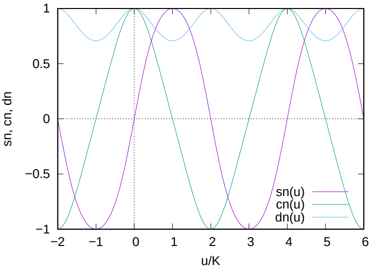
周期性

$$K = K(k) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}$$ 第 1 種完全楕円積分.

$$\mathrm{sn}(u+K) = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}, \quad \mathrm{cn}(u+K) = -k'\frac{\mathrm{sn}\, u}{\mathrm{dn}\, u}, \quad \mathrm{dn}(u+K) = \frac{k'}{\mathrm{dn}\, u},$$

$$\mathrm{sn}(u+2K) = -\mathrm{sn}\, u, \quad \mathrm{cn}(u+2K) = -\mathrm{cn}\, u, \quad \mathrm{dn}(u+2K) = \mathrm{dn},$$

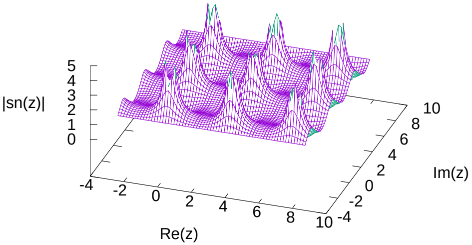
$$\mathrm{sn}(u+4K) = \mathrm{sn}\, u, \quad \mathrm{cn}(u+4K) = \mathrm{cn}\, u.$$



今回の予定

- Jacobi の楕円関数 sn, cn, dn を複素数変数の関数に拡張する.
- sn, cn, dn は**二重周期関数**である, i.e.
  複素平面の 2 方向に周期を持つ.

$$\mathrm{sn}(z+4K)=\mathrm{sn}\,z, \quad \mathrm{sn}(z+2\mathrm{i}K'), \quad \text{etc.}$$



$|\mathrm{sn}\,z|$ のグラフ.

複素関数への拡張

まず，sn, cn, dn の変数を純虚数に拡張する．

$$x = \mathrm{sn}(u; k) \quad \Leftrightarrow \quad u = \int_0^x \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

$u = \mathrm{i}v$, $x = \mathrm{i}y$ とおく．

$$x = \mathrm{sn}(\mathrm{i}v; k), \quad v = \int_0^y \frac{\mathrm{d}y}{\sqrt{(1+y^2)(1+k^2y^2)}}.$$

$y = \tan\psi$ とおくと，

$$\mathrm{d}y = \frac{\mathrm{d}\psi}{\cos^2\psi}, \quad 1 + y^2 = \frac{1}{\cos^2\psi}, \quad 1 + k^2y^2 = \frac{1 - k'^2\sin^2\psi}{\cos^2\psi}$$

$$(\, k' = \sqrt{1-k^2}\,),$$

$$v = \int_0^\psi \frac{\mathrm{d}\psi}{\sqrt{1 - k'^2\sin^2\psi}}, \quad \text{i.e.} \quad \psi = \mathrm{am}(v; k') \quad \text{振幅関数}.$$

複素関数への拡張

$$\sin\psi = \mathrm{sn}(v;k'), \quad \cos\psi = \mathrm{cn}(v;k'),$$
$$y = \tan\psi = \frac{\mathrm{sn}(v;k')}{\mathrm{cn}(v;k')},$$

$x = \mathrm{sn}(\mathrm{i}v;k) = \mathrm{i}y$ であったから，

$$\mathrm{sn}(\mathrm{i}v;k) = \mathrm{i}\frac{\mathrm{sn}(v;k')}{\mathrm{cn}(v;k')} \quad \left(k' = \sqrt{1-k^2}\right).$$

$\mathrm{cn}\,u = \sqrt{1-\mathrm{sn}^2 u}$, $\mathrm{dn}\,u = \sqrt{1-k^2\,\mathrm{sn}^2 u}$ に代入して，

$$\mathrm{cn}(\mathrm{i}v;k) = \frac{1}{\mathrm{cn}(v;k')}, \quad \mathrm{dn}(\mathrm{i}v;k) = \frac{\mathrm{dn}(v;k')}{\mathrm{cn}(v;k')}.$$

複素関数への拡張

sn, cn, dn の変数を全複素数に拡張する．

sn の加法定理で $v \to \mathrm{i}v$ とおくことにより，

$$
\begin{aligned}
\mathrm{sn}(u+\mathrm{i}v;k) &= \frac{\mathrm{sn}(u;k)\,\mathrm{cn}(\mathrm{i}v;k)\,\mathrm{dn}(\mathrm{i}v;k)+\mathrm{sn}(\mathrm{i}v;k)\,\mathrm{cn}(u;k)\,\mathrm{dn}(u;k)}{1-k^2\,\mathrm{sn}^2(u;k)\,\mathrm{sn}^2(\mathrm{i}v;k)} \\
&= \frac{\mathrm{sn}(u;k)\dfrac{1}{\mathrm{cn}(v;k')}\dfrac{\mathrm{dn}(v;k')}{\mathrm{cn}(v;k')}+\mathrm{i}\dfrac{\mathrm{sn}(v;k')}{\mathrm{cn}(v;k')}\,\mathrm{cn}(u;k)\,\mathrm{dn}(u;k)}{1+k^2\,\mathrm{sn}^2(u;k)\dfrac{\mathrm{sn}^2(v;k')}{\mathrm{cn}^2(v;k')}} \\
&= \frac{\mathrm{sn}(u;k)\,\mathrm{dn}(v;k')+\mathrm{i}\,\mathrm{cn}(u;k)\,\mathrm{dn}(u;k)\,\mathrm{sn}(v;k')\,\mathrm{cn}(v;k')}{\mathrm{cn}^2(v;k')+k^2\,\mathrm{sn}^2(u;k)\,\mathrm{sn}^2(v;k')}.
\end{aligned}
$$

cn, dn についても同様の計算を行う．

複素関数への拡張

複素関数としての sn, cn, dn

$$\operatorname{sn}(u+\mathrm{i}v;k)=\frac{\operatorname{sn}(u;k)\operatorname{dn}(v;k')+\mathrm{i}\operatorname{cn}(u;k)\operatorname{dn}(u;k)\operatorname{sn}(v;k')\operatorname{cn}(v;k')}{\operatorname{cn}^2(v;k')+k^2\operatorname{sn}^2(u;k)\operatorname{sn}^2(v;k')},$$

$$\operatorname{cn}(u+\mathrm{i}v;k)=\frac{\operatorname{cn}(u;k)\operatorname{cn}(v;k')-\mathrm{i}\operatorname{sn}(u;k)\operatorname{dn}(u;k)\operatorname{sn}(v;k')\operatorname{dn}(v;k')}{\operatorname{cn}^2(v;k')+k^2\operatorname{sn}^2(u;k)\operatorname{sn}^2(v;k')},$$

$$\operatorname{dn}(u+\mathrm{i}v;k)=\frac{\operatorname{dn}(u;k)\operatorname{cn}(v;k')\operatorname{dn}(v;k')-\mathrm{i}k^2\operatorname{sn}(u;k)\operatorname{cn}(u;k)\operatorname{sn}(v;k')}{\operatorname{cn}^2(v;k')+k^2\operatorname{sn}^2(u;k)\operatorname{sn}^2(v;k')}$$

$$(\,k'=\sqrt{1-k^2}\,).$$

複素関数への拡張

とくに $v = K'$ ($K' = K(k')$) とおいて,

$$\mathrm{sn}(u + \mathrm{i}K'; k) = \frac{1}{k\,\mathrm{sn}(u; k)}, \quad \mathrm{cn}(u + \mathrm{i}K'; k) = -\frac{\mathrm{i}}{k}\frac{\mathrm{dn}(u; k)}{\mathrm{sn}(u; k)},$$

$$\mathrm{dn}(u + \mathrm{i}K'; k) = -\mathrm{i}\frac{\mathrm{cn}(u; k)}{\mathrm{sn}(u; k)}.$$

$$\mathrm{sn}(u + 2\mathrm{i}K'; k) = \mathrm{sn}(u; k), \quad \mathrm{cn}(u + 2\mathrm{i}K'; k) = -\mathrm{cn}(u; k),$$

$$\mathrm{dn}(u + 2\mathrm{i}K'; k) = -\mathrm{dn}(u; k).$$

$$\mathrm{cn}(u + 2K + 2\mathrm{i}K'; k) = \mathrm{cn}(u; k), \quad \mathrm{dn}(u + 4\mathrm{i}K'; k) = \mathrm{dn}(u; k).$$

$\mathrm{sn}(u; k)$ は $2\mathrm{i}K'$ を, $\mathrm{cn}(u; k)$ は $2K + 2\mathrm{i}K'$ を, $\mathrm{dn}(u; k)$ は $4\mathrm{i}K'$ を周期にもつ.

二重周期性，零点・特異点

sn, cn, dn の二重周期性

sn, cn, dn は二重周期関数である，i.e.,
複素平面内の 2 方向に周期を持つ．

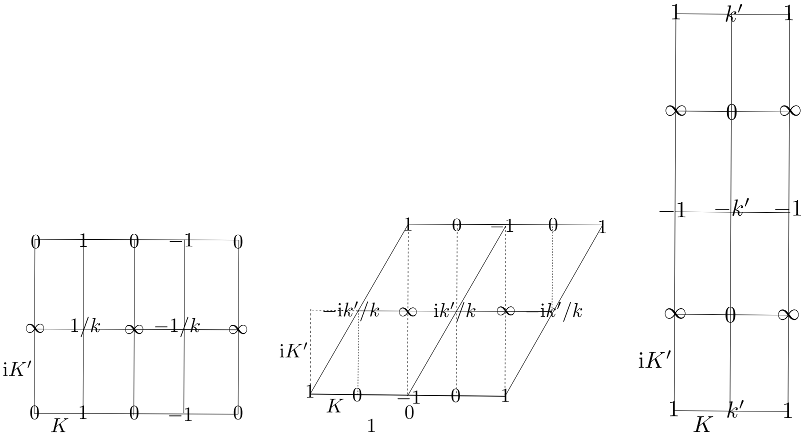
	周期
$\mathrm{sn}(u;k)$	$4K, 2\mathrm{i}K'$
$\mathrm{cn}(u;k)$	$4K, 2K+2\mathrm{i}K'$
$\mathrm{dn}(u;k)$	$2K, 4\mathrm{i}K'$

sn, cn, dn の零点・特異点

	零点	特異点
sn	$2mK+2\mathrm{i}nK'$	$2mK+(2n+1)\mathrm{i}K'$
cn	$(2m+1)K+2\mathrm{i}nK'$	$2mK+(2n+1)\mathrm{i}K'$
dn	$(2m+1)K+(2n+1)\mathrm{i}K'$	$2mK+(2n+1)\mathrm{i}K'$

（m, n は整数）

sn, cn, dn の零点・特異点



sn　　cn　　dn

sn, cn, dn の値.

sn, cn, dn は解析関数である

$z = u + \mathrm{i}v$ とおく.

sn z, cn z, dn z は解析関数である
（特異点 $2mK + (2n+1)\mathrm{i}K'$ （m, n は整数）を除いて）.

つまり,

$$f(u, v) = \operatorname{Re} \operatorname{sn}(u + \mathrm{i}v, k), \quad g(u, v) = \operatorname{Im} \operatorname{sn}(u + \mathrm{i}v; k)$$

は Cauchy-Riemann の関係式を満たす：

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v}, \quad \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u}.$$

cn, dn についても同様である.

sn, cn, dn は解析関数である

まず，実関数としての sn, cn, dn の導関数を求める．

$x = \operatorname{sn} u$ とおくと，

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \sqrt{(1-x^2)(1-k^2x^2)},$$
$$\operatorname{cn} u = \sqrt{1-\operatorname{sn}^2 u}, \quad \operatorname{dn} u = \sqrt{1-k^2\operatorname{sn}^2 u}$$

であるから，

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u.$$

$$\operatorname{cn}^2 u = 1-\operatorname{sn}^2 u, \quad \operatorname{dn}^2 u = 1-k^2\operatorname{sn}^2 u$$

の両辺を微分することにより，

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \quad \frac{\mathrm{d}}{\mathrm{d}u}\operatorname{dn} u = -k^2\operatorname{sn} u \operatorname{cn} u.$$

sn, cn, dn は解析関数である

$\mathrm{sn}(u+\mathrm{i}v;k)$ が Cauchy-Riemann 関係式を満たすことの証明（結構面倒…）．
簡単のため，$\mathrm{sn}\,u=\mathrm{sn}(u;k)$, $\overline{\mathrm{sn}}v=\mathrm{sn}(v;k')$ などと記す．

$$f(u,v)\equiv \mathrm{Re}\,\mathrm{sn}(u+\mathrm{i}v;k)=\frac{\mathrm{sn}\,u\overline{\mathrm{dn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v},$$

$$g(u,v)\equiv \mathrm{Im}\,\mathrm{sn}(u+\mathrm{i}v;k)=\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v}$$

$$(\,\overline{\mathrm{cn}}^2v+k^2\,\mathrm{sn}^2\,u\overline{\mathrm{sn}}^2v=\overline{\mathrm{cn}}^2v+(1-\mathrm{dn}^2\,u)\overline{\mathrm{sn}}^2v=1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v \quad \text{に注意}\,).$$

$$\begin{aligned}
\frac{\partial f}{\partial u}&=\frac{(\mathrm{sn}\,u)'\overline{\mathrm{dn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v}-\frac{\mathrm{sn}\,u\overline{\mathrm{dn}}v(-\mathrm{dn}^2\,u)'\overline{\mathrm{sn}}^2v}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2}\\
&=\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u\overline{\mathrm{dn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v}+\frac{\mathrm{sn}\,u\overline{\mathrm{dn}}v\cdot 2(-k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u)\,\mathrm{dn}\,u\overline{\mathrm{sn}}^2v}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2}\\
&=\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u\overline{\mathrm{dn}}v(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v-2k^2\,\mathrm{sn}^2\,u\overline{\mathrm{sn}}^2v)}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2}\\
&=\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u\overline{\mathrm{dn}}v(\overline{\mathrm{cn}}^2v-k^2\,\mathrm{sn}^2\,u\overline{\mathrm{sn}}^2v)}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2},
\end{aligned}$$

sn, cn, dn は解析関数である

$$
\begin{aligned}
\frac{\partial g}{\partial v} &= \frac{\operatorname{cn} u \operatorname{dn} u (\overline{\operatorname{sn}} v)' \overline{\operatorname{cn}} v + \operatorname{cn} u \operatorname{dn} u \overline{\operatorname{sn}} v (\overline{\operatorname{cn}} v)'}{1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} + \frac{\operatorname{cn} u \operatorname{dn} u \overline{\operatorname{sn}} v \overline{\operatorname{cn}} v \cdot \operatorname{dn}^2 u (\overline{\operatorname{sn}}^2 v)'}{(1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2} \\
&= \frac{\operatorname{cn} u \operatorname{dn} u \overline{\operatorname{cn}}^2 v \overline{\operatorname{dn}} v - \operatorname{cn} u \operatorname{dn} u \overline{\operatorname{sn}}^2 v \overline{\operatorname{dn}} v}{1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} + \frac{2 \operatorname{cn} u \operatorname{dn}^3 u \overline{\operatorname{sn}} v \overline{\operatorname{cn}} v \cdot \overline{\operatorname{sn}} v \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{(1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2} \\
&= \frac{\operatorname{cn} u \operatorname{dn} u \overline{\operatorname{dn}} v \left[(\overline{\operatorname{cn}}^2 v - \overline{\operatorname{sn}}^2 v)((1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v) + 2 \operatorname{dn}^2 \overline{\operatorname{sn}}^2 v \overline{\operatorname{cn}}^2 v \right]}{(1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2},
\end{aligned}
$$

$$
\begin{aligned}
\text{分子の } [\cdots] &= (2\overline{\operatorname{cn}}^2 v - 1)(1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v) + 2 \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v \overline{\operatorname{cn}}^2 v \\
&= 2\overline{\operatorname{cn}}^2 v - 1 + \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v \\
&= \overline{\operatorname{cn}}^2 v - \overline{\operatorname{sn}}^2 v + \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v \\
&= \overline{\operatorname{cn}}^2 v - k^2 \operatorname{sn}^2 u \overline{\operatorname{sn}}^2 v,
\end{aligned}
$$

$$
\therefore \quad \frac{\partial g}{\partial v} = \frac{\operatorname{cn} u \operatorname{dn} u \overline{\operatorname{dn}} v (\overline{\operatorname{cn}}^2 v - k^2 \operatorname{sn}^2 u \overline{\operatorname{sn}}^2 v)}{(1 - \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2} = \frac{\partial f}{\partial u}.
$$

sn, cn, dn は解析関数である

$$\frac{\partial f}{\partial v} = \frac{\operatorname{sn} u(\overline{\operatorname{dn} v})'}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + \frac{\operatorname{sn} u\overline{\operatorname{dn}} v\operatorname{dn}^2 u(\overline{\operatorname{sn}}^2 v)'}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}$$

$$= -\frac{k'^2 \operatorname{sn} u\overline{\operatorname{sn}} v\overline{\operatorname{cn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + \frac{2\operatorname{sn} u\overline{\operatorname{dn}} v\operatorname{dn}^2 u\overline{\operatorname{sn}} v\overline{\operatorname{cn}} v\overline{\operatorname{dn}} v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}$$

$$= \frac{\operatorname{sn} u\overline{\operatorname{sn}} v\overline{\operatorname{cn}} v\left[-k'^2(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)+2\operatorname{dn}^2 u\overline{\operatorname{dn}}^2 v\right]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2},$$

$$\text{分子の } [\cdots] = k'^2\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v - k'^2 + 2\operatorname{dn}^2 u\overline{\operatorname{dn}}^2 v$$

$$= \operatorname{dn}^2 u(1+\overline{\operatorname{dn}}^2 v) - k'^2,$$

$$\frac{\partial f}{\partial v} = \frac{\operatorname{sn} u\overline{\operatorname{sn}} v\overline{\operatorname{cn}} v\left[\operatorname{dn}^2 u(1+\overline{\operatorname{dn}}^2 v)-k'^2\right]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}.$$

sn, cn, dn は解析関数である

$$\frac{\partial g}{\partial u} = \frac{(\mathrm{cn}\,u)'\,\mathrm{dn}\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v + \mathrm{cn}\,u(\mathrm{dn}\,u)'\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v} + \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v(\mathrm{dn}^2\,u)'\overline{\mathrm{sn}}^2v}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2}$$

$$= \frac{-\mathrm{sn}\,u\,\mathrm{dn}^2\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v - k^2\,\mathrm{sn}\,u\,\mathrm{cn}^2\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v}{1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v} - \frac{2k^2\,\mathrm{sn}\,u\,\mathrm{cn}^2\,u\,\mathrm{dn}^2\,u\overline{\mathrm{sn}}^3v\overline{\mathrm{cn}}v}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2}$$

$$= -\frac{\mathrm{sn}\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v\left[(\mathrm{dn}^2\,u + k^2\,\mathrm{cn}^2\,u)(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v) + 2k^2\,\mathrm{cn}^2\,u\,\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v\right]}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2},$$

$$\text{分子の }[\cdots] = \mathrm{dn}^2\,u + k^2\,\mathrm{cn}^2\,u - \mathrm{dn}^4\,u\overline{\mathrm{sn}}^2v + k^2\,\mathrm{cn}^2\,u\,\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v$$

$$= -k'^2 + 2\,\mathrm{dn}^2\,u + \mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v(k^2\,\mathrm{cn}^2\,u - \mathrm{dn}^2\,u)$$

$$= -k'^2 + 2\,\mathrm{dn}^2\,u - k'^2\,\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v$$

$$= \mathrm{dn}^2\,u(1+\overline{\mathrm{dn}}^2v) - k'^2,$$

$$\therefore \quad \frac{\partial g}{\partial u} = -\frac{\mathrm{sn}\,u\overline{\mathrm{sn}}v\overline{\mathrm{cn}}v[\mathrm{dn}^2\,u(1+\overline{\mathrm{dn}}^2v) - k'^2]}{(1-\mathrm{dn}^2\,u\overline{\mathrm{sn}}^2v)^2} = -\frac{\partial f}{\partial v}.$$

ゆえに，$f(u,v) = \mathrm{Re}\,\mathrm{sn}(u+\mathrm{i}v;k)$, $g(u,v) = \mathrm{Im}\,\mathrm{sn}(u+\mathrm{i}v;k)$ は Cauchy-Riemann 関係式を満たす.

cn, dn についても，Cauchy-Riemann 関係式の成立が示される（補遺参照）.

sn, cn, dn の性質

- $\mathrm{sn}(z;k)$, $\mathrm{cn}(z;k)$, $\mathrm{dn}(z;k)$ は全複素平面 $\mathbb{C}$ における有理型関数である.

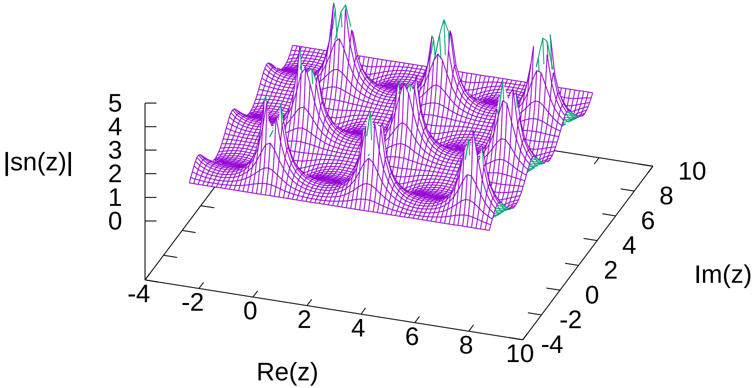
  ∗ 有理型関数：極を除いて解析的な関数.

- $$\begin{aligned}\mathrm{sn}'(z;k) &= \mathrm{cn}(z;k)\,\mathrm{dn}(z;k),\\ \mathrm{cn}'(z;k) &= -\,\mathrm{sn}(z;k)\,\mathrm{dn}(z;k),\\ \mathrm{dn}'(z;k) &= -\,k^2\,\mathrm{sn}(z;k)\,\mathrm{cn}(z;k).\end{aligned}$$

- $\mathrm{sn}(z;k)$, $\mathrm{cn}(z;k)$, $\mathrm{dn}(z;k)$ は**二重周期関数**である, i.e., 複素平面内の 2 方向に周期を持つ.

	周期
$\mathrm{sn}(z;k)$	$4K, 2\mathrm{i}K'$
$\mathrm{cn}(z;k)$	$4K, 2K+2\mathrm{i}K'$
$\mathrm{dn}(z;k)$	$2K, 4\mathrm{i}K'$

sn, cn, dn の性質



$|\operatorname{sn}(z;k)|$ のグラフ ($k = \sqrt{0.5}$).

sn, cn, dn の性質

- 零点と極（m, n は整数）

	零点	極
$\mathrm{sn}(z;k)$	$2mK+2\mathrm{i}nK'$	$2mK+(2n+1)\mathrm{i}K'$
$\mathrm{cn}(z;k)$	$(2m+1)K+2\mathrm{i}nK'$	$2mK+(2n+1)\mathrm{i}K'$
$\mathrm{dn}(z;k)$	$(2m+1)K+(2n+1)\mathrm{i}K'$	$2mK+(2n+1)\mathrm{i}K'$

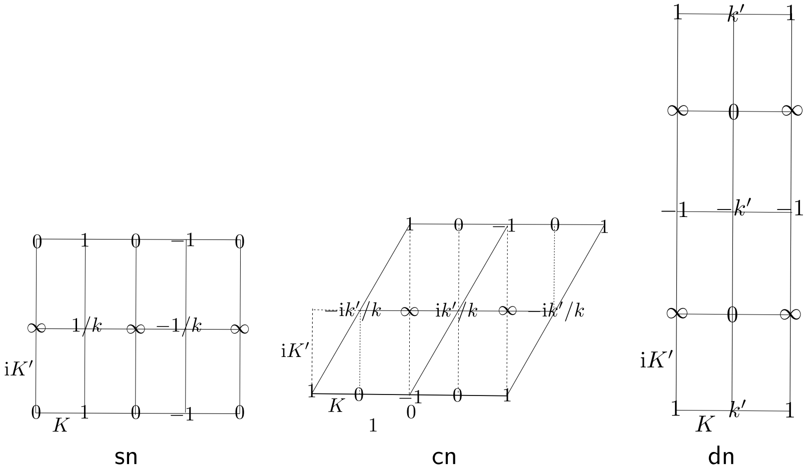
これらの零点・極はすべて **1 位**である．

上記の零点が 1 位であること．
sn について，$(\mathrm{sn}\, z)' = \mathrm{cn}\, z\, \mathrm{dn}\, z$ より，零点 $z = 2mK + 2\mathrm{i}nK'$ において $(\mathrm{sn}\, z)' \neq 0$, etc.

上記の極が 1 位であること．

sn について，$\mathrm{sn}(z+\mathrm{i}K') = 1/(k\, \mathrm{sn}\, z)$ に注意すれば，sn の極 $z = 2mK + (2n+1)\mathrm{i}K'$ は 1 位であることがわかる，etc.

sn, cn, dn の性質



sn, cn, dn の値.

楕円関数の一般的定義

sn, cn, dn の重要な性質

sn z, cn z, dn z は全複素平面で有理型である二重周期関数である.

楕円関数の一般的定義

sn, cn, dn の重要な性質

sn z, cn z, dn z は全複素平面で有理型である二重周期関数である.

楕円関数の定義

全複素平面 $\mathbb{C}$ で有理型である二重周期関数を楕円関数とよぶ.

sn, cn, dn は楕円関数の例である.

次回は，楕円関数の一般論について学びます.

（補遺）cn に対して Cauchy-Riemann 関係式を確認

$$f(u,v) = \operatorname{Re}\operatorname{cn}(u+\mathrm{i}v;k) = \frac{\operatorname{cn} u\overline{\operatorname{cn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v},$$

$$g(u,v) = \operatorname{Im}\operatorname{cn}(u+\mathrm{i}v;k) = -\frac{\operatorname{sn} u \operatorname{dn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v}.$$

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{(\operatorname{cn} u)'\overline{\operatorname{cn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + \frac{\operatorname{cn} u\overline{\operatorname{cn}} v(\operatorname{dn}^2 u)'\overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}\\ &= -\frac{\operatorname{sn} u \operatorname{dn} u\overline{\operatorname{cn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} - 2k^2\frac{\operatorname{cn} u\overline{\operatorname{cn}} v \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u\overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}\\ &= -\frac{\operatorname{sn} u \operatorname{dn} u\overline{\operatorname{cn}} v(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v + 2k^2\operatorname{cn}^2 u\overline{\operatorname{sn}}^2 v)}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2},\end{aligned}$$

$$\begin{aligned}\text{分子の } (\cdots) &= 1-(1-k^2\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v + 2k^2(1-\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v\\ &= 1+(2k^2-1)\overline{\operatorname{sn}}^2 v - k^2\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v,\end{aligned}$$

$$\therefore\quad \frac{\partial f}{\partial u} = -\frac{\operatorname{sn} u \operatorname{dn} u\overline{\operatorname{cn}} v[1+(2k^2-1)\overline{\operatorname{sn}}^2 v - k^2\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}.$$

（補遺）cn に対して Cauchy-Riemann 関係式を確認

$$\frac{\partial g}{\partial v} = -\frac{\operatorname{sn} u \operatorname{dn} u[(\overline{\operatorname{sn}} v)'\overline{\operatorname{dn}} v + \overline{\operatorname{sn}} v(\overline{\operatorname{dn}} v)']}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} - \frac{\operatorname{sn} u \operatorname{dn} u \overline{\operatorname{sn}} v \overline{\operatorname{dn}} v \operatorname{dn}^2 u(\overline{\operatorname{sn}}^2 v)'}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -\frac{\operatorname{sn} u \operatorname{dn} u[\overline{\operatorname{cn}} v \overline{\operatorname{dn}}^2 v - k'^2 \overline{\operatorname{sn}}^2 v \overline{\operatorname{cn}} v]}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} - \frac{2 \operatorname{sn} u \operatorname{dn} u \overline{\operatorname{sn}} v \overline{\operatorname{dn}} v \operatorname{dn}^2 u \overline{\operatorname{sn}} v \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -\frac{\operatorname{sn} u \operatorname{dn} u \overline{\operatorname{cn}} v[(1-2k'^2 \overline{\operatorname{sn}}^2 v)(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v) + 2\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v \overline{\operatorname{dn}}^2 v]}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2},$$

$$\text{分子の } [\cdots] = (1-2k'^2 \overline{\operatorname{sn}}^2 v)(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v) + 2\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v(1-k'^2 \overline{\operatorname{sn}}^2 v)$$

$$= 1 - 2k'^2 \overline{\operatorname{sn}}^2 v + \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v$$

$$= 1 + (2k^2-1)\overline{\operatorname{sn}}^2 v - k^2 \operatorname{sn}^2 u \overline{\operatorname{sn}}^2 v,$$

$$\therefore \quad \frac{\partial g}{\partial v} = -\frac{\operatorname{sn} u \operatorname{dn} u \overline{\operatorname{cn}} v[1+(2k^2-1)\overline{\operatorname{sn}}^2 v - k^2 \operatorname{sn}^2 u \overline{\operatorname{sn}}^2 v]}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2} = \frac{\partial f}{\partial u}.$$

（補遺）cn に対して Cauchy-Riemann 関係式を確認

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\operatorname{cn} u(\overline{\operatorname{cn}} v)'}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + \frac{\operatorname{cn} u\overline{\operatorname{cn}} v \operatorname{dn}^2 u(\overline{\operatorname{sn}}^2 v)'}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} \\ &= -\frac{\operatorname{cn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 \overline{\operatorname{sn}}^2 v} + \frac{2\operatorname{cn} u\overline{\operatorname{cn}} v \operatorname{dn}^2 u\overline{\operatorname{sn}} v\overline{\operatorname{cn}} v\overline{\operatorname{dn}} v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} \\ &= \frac{\operatorname{cn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v[-(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)+2\operatorname{dn}^2 u\overline{\operatorname{cn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} \\ &= \frac{\operatorname{cn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v(-1+2\operatorname{dn}^2 u-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial u} &= -\frac{[(\operatorname{sn} u)'\operatorname{dn} u+\operatorname{sn} u(\operatorname{dn} u)']\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} - \frac{\operatorname{sn} u\operatorname{dn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v(\operatorname{dn}^2 u)'\overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} \\ &= -\frac{\operatorname{cn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v(\operatorname{dn}^2 u-k^2\operatorname{sn}^2 u)}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + 2k^2\frac{\operatorname{sn} u\operatorname{dn} u\overline{\operatorname{sn}}^3 v\overline{\operatorname{dn}} v\operatorname{sn} u\operatorname{cn} u\operatorname{dn} u}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} \\ &= \frac{\operatorname{cn} u\overline{\operatorname{sn}} v\overline{\operatorname{dn}} v[-(2\operatorname{dn}^2 u-1)(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)+2k^2\operatorname{sn}^2 u\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2},\end{aligned}$$

（補遺）cn に対して Cauchy-Riemann 関係式を確認

$$
\begin{aligned}
\text{分子の } [\cdots] &= -2\,\mathrm{dn}^2 u + 1 + 2\,\mathrm{dn}^4 u\overline{\mathrm{sn}}^2 v - \mathrm{dn}^2 u\overline{\mathrm{sn}}^2 v + 2(1-\mathrm{dn}^2 u)\,\mathrm{dn}^2 u\overline{\mathrm{sn}}^2 v \\
&= 1 - 2\,\mathrm{dn}^2 u + \mathrm{dn}^2 u\overline{\mathrm{sn}}^2 v, \\
\therefore \quad \frac{\partial g}{\partial u} &= \frac{\mathrm{cn}\, u\overline{\mathrm{sn}}\, v\overline{\mathrm{dn}}\, v(1-2\,\mathrm{dn}^2 u + \mathrm{dn}^2 u\overline{\mathrm{sn}}^2 v)}{(1-\mathrm{dn}^2 u\overline{\mathrm{sn}}^2 v)^2} = -\frac{\partial f}{\partial v}.
\end{aligned}
$$

以上より，$\mathrm{cn}(u+\mathrm{i}v;k)$ の実部・虚部は Cauchy-Riemann 関係式を満たす.

（補遺）dn に対して Cauchy-Riemann 関係式を確認

$$f(u,v) = \operatorname{Re}\operatorname{dn}(u+\mathrm{i}v;k) = \frac{\operatorname{dn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v},$$

$$g(u,v) = \operatorname{Im}\operatorname{dn}(u+\mathrm{i}v;k) = -k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{sn}} v}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v}.$$

$$\frac{\partial f}{\partial u} = \frac{(\operatorname{dn} u)' \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} + \frac{\operatorname{dn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v (\operatorname{dn}^2 u)' \overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} - 2k^2 \frac{\operatorname{sn} u \operatorname{cn} u \operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v (1+\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}.$$

$$\frac{\partial g}{\partial v} = -k^2 \frac{\operatorname{sn} u \operatorname{cn} u (\overline{\operatorname{sn}} v)'}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} - k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{sn}} v \operatorname{dn}^2 u (\overline{\operatorname{sn}}^2 v)'}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v} - 2k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{sn}} v \operatorname{dn}^2 u \overline{\operatorname{sn}} v \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2}$$

$$= -k^2 \frac{\operatorname{sn} u \operatorname{cn} u \overline{\operatorname{cn}} v \overline{\operatorname{dn}} v (1+\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)}{(1-\operatorname{dn}^2 u \overline{\operatorname{sn}}^2 v)^2} = \frac{\partial f}{\partial u}.$$

（補遺）dn に対して Cauchy-Riemann 関係式を確認

$$\frac{\partial f}{\partial v} = \frac{\operatorname{dn} u[(\overline{\operatorname{cn}}v)'\overline{\operatorname{dn}}v + \overline{\operatorname{cn}}v(\overline{\operatorname{dn}}v)']}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + \frac{\operatorname{dn} u\overline{\operatorname{cn}}v\overline{\operatorname{dn}}v \operatorname{dn}^2 u(\overline{\operatorname{sn}}^2 v)'}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}$$

$$= \frac{\operatorname{dn} u[\overline{\operatorname{sn}}v\overline{\operatorname{dn}}^2 v + k'^2\overline{\operatorname{sn}}v\overline{\operatorname{cn}}^2 v]}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v} + 2\frac{\operatorname{dn} u^3\overline{\operatorname{sn}}v\overline{\operatorname{cn}}^2 v\overline{\operatorname{dn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}$$

$$= -\frac{\operatorname{dn} u\overline{\operatorname{sn}}v[-(1+k'^2-2k'^2\overline{\operatorname{sn}}^2 v)(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v) + 2\operatorname{dn}^2 u\overline{\operatorname{cn}}^2 v\overline{\operatorname{dn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2},$$

分子の $[\cdots]$

$$\begin{aligned} &= -(1+k'^2-2k'^2\overline{\operatorname{sn}}^2 v)(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v) \\ &\quad + 2\operatorname{dn}^2 u(1-\overline{\operatorname{sn}}^2 v)(1-k'^2\overline{\operatorname{sn}}^2 v) \\ &= -1-k'^2-(1+k'^2)\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v + 2k'^2\overline{\operatorname{sn}}^2 v + 2\operatorname{dn}^2 u \\ &= -2+k^2-(2-k^2)(1-k^2\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v + 2(1-k^2)\overline{\operatorname{sn}}^2 v \\ &\quad + 2(1-k^2\operatorname{sn}^2 u) \\ &= k^2[1-2\operatorname{sn}^2 u - \overline{\operatorname{sn}}^2 v + (2-k^2)\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v] \end{aligned}$$

$$\therefore \quad \frac{\partial f}{\partial v} = k^2\frac{1-2\operatorname{sn}^2 u - \overline{\operatorname{sn}}^2 v + (2-k^2)\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2}.$$

（補遺）dn に対して Cauchy-Riemann 関係式を確認

$$\frac{\partial g}{\partial u} = -k^2\frac{[(\operatorname{sn} u)'\operatorname{cn} u + \operatorname{sn} u(\operatorname{cn} u)']\overline{\operatorname{sn}}v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}v} - k^2\frac{\operatorname{sn} u\operatorname{cn} u\overline{\operatorname{sn}}v(\operatorname{dn}^2 u)'\overline{\operatorname{sn}}^2 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}v)^2}$$

$$= -k^2\frac{(\operatorname{cn}^2 u\operatorname{dn} u - \operatorname{sn}^2 u\operatorname{dn} u)\overline{\operatorname{sn}}v}{1-\operatorname{dn}^2 u\overline{\operatorname{sn}}v} + 2k^4\frac{\operatorname{sn}^2 u\operatorname{cn}^2 u\operatorname{dn} u\overline{\operatorname{sn}}^3 v}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}v)^2}$$

$$= -k^2\frac{\operatorname{dn} u\overline{\operatorname{sn}}v[(1-2\operatorname{sn}^2 u)(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v) - 2k^2\operatorname{sn}^2 u\operatorname{cn}^2 u\overline{\operatorname{sn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}v)^2},$$

分子の $[\cdots] = 1-2\operatorname{sn}^2 u - \operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v + 2\operatorname{sn}^2 u\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v - 2k^2\operatorname{sn}^2 u\operatorname{cn}^2 u\overline{\operatorname{sn}}^2 v$

$$= 1-2\operatorname{sn}^2 u - (1-k^2\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v + 2\operatorname{sn}^2 u(1-k^2\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v - 2k^2\operatorname{sn}^2 u(1-\operatorname{sn}^2 u)\overline{\operatorname{sn}}^2 v$$

$$= 1-2\operatorname{sn}^2 u - \overline{\operatorname{sn}}^2 v + (2-k^2)\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v,$$

$$\therefore\quad \frac{\partial g}{\partial u} = -k^2\frac{\operatorname{dn} u\overline{\operatorname{sn}}v[1-2\operatorname{sn}^2 u - \overline{\operatorname{sn}}^2 v + (2-k^2)\operatorname{sn}^2 u\overline{\operatorname{sn}}^2 v]}{(1-\operatorname{dn}^2 u\overline{\operatorname{sn}}^2 v)^2} = -\frac{\partial f}{\partial v}.$$

以上より，$\operatorname{dn}(u+\mathrm{i}v;k)$ の実部・虚部に対して Cauchy-Riemann 関係式が示された．