解析力学 (5)・Hamilton 力学（続）・補遺

幾何学的視点から

緒方秀教

電気通信大学大学院　情報・ネットワーク工学専攻

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Poisson 括弧

Poisson 括弧の性質

1. **双線形性．**
$$\begin{aligned}\{c_1A_1+c_2A_2,B\}&=c_1\{A_1,B\}+c_2\{A_2,B\}\\ \{A,c_1B_1+c_2B_2\}&=c_1\{A,B_1\}+c_2\{A,B_2\}\end{aligned}\quad (\,c_1,c_2\ :\ \text{const.}\,).$$
2. **反可換性．**
$$\{A,B\}=-\{B,A\}.$$
3. **Jacobi 律．**
$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0.$$

1., 2. を示すのは容易．

3. は直接計算でも示せるが結構面倒である．

井田大輔「現代解析力学入門」（朝倉書店，2020 年）にうまい証明があるので，これを紹介する．

Poisson 括弧：Jacobi 律の証明

まず，記号・言葉の準備．

$$T^{\mu\nu} \text{ の対称部分} \quad T^{(\mu\nu)} := \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}), \quad \text{反対称部分} \quad T^{[\mu\nu]} := \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}),$$
$$T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]}.$$

一般に次が成り立つ．

$$T^{(\mu\nu)} S_{\mu\nu} = T^{(\mu\nu)} S_{(\mu\nu)}, \quad T^{[\mu\nu]} S_{\mu\nu} = T^{[\mu\nu]} S_{[\mu\nu]}, \text{ etc.}$$ （反）対称性は「遺伝」する．

$$\because \quad T^{(\mu\nu)} S_{\mu\nu} = T^{(\mu\nu)} (S_{(\mu\nu)} + S_{[\mu\nu]}) = T^{(\mu\nu)} S_{(\mu\nu)} + \underbrace{T^{(\mu\nu)} S_{[\mu\nu]}}_{0} = T^{(\mu\nu)} S_{(\mu\nu)}.$$

次の記法を用いる：$\partial_\mu = \dfrac{\partial}{\partial z^\mu}$.

Poisson 括弧：Jacobi 律の証明

$$
\begin{aligned}
\{A,\{B,C\}\} &= J^{\mu\nu}(\partial_\mu A)\partial_\nu\left[J^{\rho\sigma}(\partial_\rho B)(\partial_\sigma C)\right] \\
&= J^{\mu\nu}J^{\rho\sigma}(\partial_\mu A)(\partial_\sigma C)(\partial_\nu\partial_\rho B) + J^{\mu\nu}J^{\rho\sigma}(\partial_\mu A)(\partial_\rho B)(\partial_\nu\partial_\sigma C).
\end{aligned}
$$

ここで，$J^{\mu\nu}J^{\rho\sigma}(\partial_\nu\partial_\rho B)$ は μ,σ について対称である．実際，

$$
J^{\mu\nu}J^{\rho\sigma}\partial_\nu\partial_\rho B = J^{\mu\nu}J^{\rho\sigma}\partial_{(\nu}\partial_{\rho)}B = J^{\mu(\nu}J^{\rho)\sigma}\partial_{(\nu}\partial_{\rho)}B
$$

であり，

$$
\begin{aligned}
J^{\mu(\nu}J^{\rho)\sigma} &= \frac{1}{2}(J^{\mu\nu}J^{\rho\sigma} + J^{\mu\rho}J^{\nu\sigma}) \\
&\quad (J^{\mu\nu} = -J^{\nu\mu} \text{ を用いて}) \\
&= \frac{1}{2}(J^{\mu\nu}J^{\rho\sigma} + J^{\rho\mu}J^{\sigma\nu}) = \frac{1}{2}(J^{\mu\nu}J^{\rho\sigma} + J^{\sigma\nu}J^{\rho\mu})
\end{aligned}
$$

であるから．よって，

$$
J^{\mu\nu}J^{\rho\sigma}(\partial_\mu A)(\partial_\sigma C)(\partial_\nu\partial_\rho B) = J^{\mu(\nu}J^{\rho)\sigma}(\partial_\mu A)(\partial_\sigma C)(\partial_{(\nu}\partial_{\rho)}B)
$$

は A,C について対称である．

Poisson 括弧：Jacobi 律の証明

同様に

$$\begin{aligned} J^{\mu\nu}J^{\rho\sigma}(\partial_\mu A)(\partial_\rho B)(\partial_\nu\partial_\sigma C) &= -J^{\mu\nu}J^{\sigma\rho}(\partial_\mu A)(\partial_\rho B)(\partial_\nu\partial_\sigma C) \\ &= -J^{\mu(\nu}J^{\sigma)\rho}(\partial_\mu A)(\partial_\rho B)(\partial_{(\nu}\partial_{\sigma)} C) \end{aligned}$$

も A, B について対称である．以上により

$$\begin{aligned} &\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\} \\ &= \underbrace{J^{\mu(\nu}J^{\rho)\sigma}(\partial_\mu A)(\partial_\sigma C)(\partial_{(\nu}\partial_{\rho)}B)}_{(1)} - \underbrace{J^{\mu(\nu}J^{\sigma)\rho}(\partial_\mu A)(\partial_\rho B)(\partial_{(\nu}\partial_{\sigma)}C)}_{(2)} + \underbrace{J^{\mu(\nu}J^{\rho)\sigma}(\partial_\mu B)(\partial_\sigma A)(\partial_{(\nu}\partial_{\rho)}C)}_{(2)} \\ &\quad - \underbrace{J^{\mu(\nu}J^{\sigma)\rho}(\partial_\mu B)(\partial_\rho C)(\partial_{(\nu}\partial_{\sigma)}A)}_{(3)} + \underbrace{J^{\mu(\nu}J^{\rho)\sigma}(\partial_\mu C)(\partial_\sigma B)(\partial_{(\nu}\partial_{\rho)}A)}_{(3)} - \underbrace{J^{\mu(\nu}J^{\sigma)\rho}(\partial_\mu C)(\partial_\rho A)(\partial_{(\nu}\partial_{\sigma)}B)}_{(1)} \end{aligned}$$

となり，同じ番号をつけた項どうし打ち消し合って 0 になる． ■

補遺：二次元 Kepler 問題，式 (3) の導出

まず，次の式を用意する．

$$x = \frac{1}{2}(\xi^2 - \eta^2), \quad y = \xi\eta, \quad \dot{x} = \xi\dot{\xi} - \eta\dot{\eta}, \quad \dot{y} = \eta\dot{\xi} + \xi\dot{\eta},$$
$$p_\xi = (\xi^2 + \eta^2)\dot{\xi}, \quad p_\eta = (\xi^2 + \eta^2)\dot{\eta}.$$

$\epsilon = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r}$ を用いて，

$$H_\xi - H_\eta = \frac{1}{2}\left\{p_\xi^2 - p_\eta^2 + \omega^2(\xi^2 - \eta^2)\right\} = \frac{1}{2}\left(p_\xi^2 - p_\eta^2 - 4\epsilon x\right) = \frac{1}{2}\left\{p_\xi^2 - p_\eta^2 - 2x(\dot{x}^2 + \dot{y}^2) + 4\mu\frac{x}{r}.\right\}.$$

$$\begin{aligned}
&p_\xi^2 - p_\eta^2 - 2x(\dot{x}^2 + \dot{y}^2) \\
&= (\xi^2 + \eta^2)^2(\dot{\xi}^2 - \dot{\eta}^2) - (\xi^2 - \eta^2)\left\{(\xi\dot{\xi} - \eta\dot{\eta})^2 + (\eta\dot{\xi} + \xi\dot{\eta})^2\right\} \\
&= 2(\xi^2 + \eta^2)(\eta^2\dot{\xi}^2 - \xi^2\dot{\eta}^2) = 2(\xi^2 + \eta^2)(\eta\dot{\xi} - \xi\dot{\eta})(\eta\dot{\xi} + \xi\dot{\eta}) \\
&\quad \left(x\dot{y} - y\dot{x} = \frac{1}{2}(\xi^2 + \eta^2)(\xi\dot{\eta} - \eta\dot{\xi} \quad \text{を用いて} \right) \\
&= -4\dot{y}(x\dot{y} - y\dot{x}) + 4\mu\frac{x}{r}.
\end{aligned}$$

よって，第 1 式を得た．

補遺：二次元 Kepler 問題，式 (3) の導出

$\epsilon = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r}$ を用いて，

$$S = p_\xi p_\eta + \omega^2 \xi\eta = p_\xi p_\eta - 2\epsilon\xi\eta = p_\xi p_\eta - y(\dot{x}^2 + \dot{y}^2) + 2\mu\frac{y}{r}.$$

$$\begin{aligned}
p_\xi p_\eta - y(\dot{x}^2 + \dot{y}^2) &= (\xi^2 + \eta^2)\left\{(\xi^2 + \eta^2)\dot{\xi}\dot{\eta} - \xi\eta(\dot{\xi}^2 + \dot{\eta}^2)\right\} \\
&= (\xi^2 + \eta^2)(\xi\dot{\xi} - \eta\dot{\eta})(\xi\dot{\eta} - \eta\dot{\xi}) \\
&\quad \left(x\dot{y} - y\dot{x} = \frac{1}{2}(\xi\dot{\eta} - \eta\dot{\xi}) \quad \text{を用いて} \right) \\
&= 2\dot{x}(x\dot{y} - y\dot{x})
\end{aligned}$$

よって第 2 式を得た． ■