# 解析力学 (8)・力学の Riemann 幾何学 (1)（補遺）

## 幾何学的視点から

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# 補遺：Christoffel の記号の変換則

Christoffel の記号 $\Gamma^{\alpha}_{\beta\gamma}$ は座標変換 $(q^{\alpha}) \to (Q^{\alpha})$ の際，次のように変換する．

$$\Gamma^{\alpha}_{\beta\gamma} \to \widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \frac{\partial q^{\lambda}}{\partial Q^{\beta}} \frac{\partial q^{\mu}}{\partial Q^{\gamma}} \Gamma^{\kappa}_{\lambda\mu} + \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \frac{\partial^2 q^{\kappa}}{\partial Q^{\beta} \partial Q^{\gamma}}.$$

【証明】座標変換後の計量テンソルを $\widetilde{g}_{\alpha\beta}$ とする．

$$\begin{aligned}
\widetilde{\Gamma}^{\alpha}_{\beta\gamma} &= \frac{1}{2} \widetilde{g}^{\alpha\delta} \left( \frac{\partial \widetilde{g}_{\delta\gamma}}{\partial Q^{\beta}} + \frac{\partial \widetilde{g}_{\delta\beta}}{\partial Q^{\gamma}} - \frac{\partial \widetilde{g}_{\beta\gamma}}{\partial q^{\delta}} \right) \\
&= \frac{1}{2} \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \frac{\partial Q^{\delta}}{\partial q^{\lambda}} g^{\kappa\lambda} \left\{ \frac{\partial}{\partial Q^{\beta}} \left( \frac{\partial q^{\mu}}{\partial Q^{\delta}} \frac{\partial q^{\nu}}{\partial Q^{\gamma}} g_{\mu\nu} \right) + \frac{\partial}{\partial Q^{\gamma}} \left( \frac{\partial q^{\mu}}{\partial Q^{\delta}} \frac{\partial q^{\nu}}{\partial Q^{\beta}} g_{\mu\nu} \right) \right. \\
&\qquad \left. - \frac{\partial}{\partial Q^{\gamma}} \left( \frac{\partial q^{\mu}}{\partial Q^{\delta}} \frac{\partial q^{\nu}}{\partial Q^{\beta}} g_{\mu\nu} \right) \right\} \\
&= \frac{1}{2} g^{\kappa\lambda} \left\{ \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \underbrace{\frac{\partial Q^{\delta}}{\partial q^{\lambda}} \frac{\partial q^{\mu}}{\partial Q^{\delta}}}_{\delta^{\mu}_{\lambda}} \frac{\partial q^{\nu}}{\partial Q^{\gamma}} \frac{\partial g_{\mu\nu}}{\partial Q^{\beta}} + \underbrace{\frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \frac{\partial Q^{\delta}}{\partial q^{\lambda}} \frac{\partial^2 q^{\mu}}{\partial Q^{\beta} \partial Q^{\delta}} \frac{\partial q^{\nu}}{\partial Q^{\gamma}} g_{\mu\nu}}_{(1)} \right. \\
&\qquad + \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \underbrace{\frac{\partial Q^{\delta}}{\partial q^{\lambda}} \frac{\partial q^{\mu}}{\partial Q^{\delta}}}_{\delta^{\mu}_{\lambda}} \frac{\partial^2 q^{\nu}}{\partial Q^{\beta} \partial Q^{\gamma}} g_{\mu\nu} + \frac{\partial Q^{\alpha}}{\partial q^{\kappa}} \underbrace{\frac{\partial Q^{\delta}}{\partial q^{\lambda}} \frac{\partial q^{\mu}}{\partial Q^{\delta}}}_{\delta^{\mu}_{\lambda}} \frac{\partial q^{\nu}}{\partial Q^{\beta}} \frac{\partial g_{\mu\nu}}{\partial Q^{\gamma}}
\end{aligned}$$

# 補遺：Christoffel の記号の変換則

$$
+\underbrace{\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial Q^\delta}{\partial q^\lambda}\frac{\partial^2 q^\mu}{\partial Q^\gamma \partial Q^\delta}\frac{\partial q^\nu}{\partial Q^\beta}g_{\mu\nu}}_{(2)}+\frac{\partial Q^\alpha}{\partial q^\kappa}\underbrace{\frac{\partial Q^\delta}{\partial q^\lambda}\frac{\partial q^\mu}{\partial Q^\delta}}_{\delta^\mu_\lambda}\frac{\partial^2 q^\nu}{\partial Q^\beta \partial Q^\gamma}g_{\mu\nu}
$$

$$
-\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\mu}{\partial Q^\beta}\frac{\partial q^\nu}{\partial Q^\gamma}\underbrace{\frac{\partial Q^\delta}{\partial q^\gamma}\frac{\partial g_{\mu\nu}}{\partial Q^\delta}}_{\frac{\partial g_{\mu\nu}}{\partial q^\lambda}}-\underbrace{\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial Q^\delta}{\partial q^\lambda}\frac{\partial^2 q^\mu}{\partial Q^\beta \partial Q^\delta}\frac{\partial q^\nu}{\partial Q^\gamma}g_{\mu\nu}}_{(1)}
$$

$$
\left.-\underbrace{\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial Q^\delta}{\partial q^\lambda}\frac{\partial q^\mu}{\partial Q^\beta}\frac{\partial^2 q^\nu}{\partial Q^\gamma \partial Q^\delta}g_{\mu\nu}}_{(2)}\right\}
$$

（同じ番号を付けた項同士が相殺しあい）

$$
=\frac{1}{2}\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\mu}{\partial Q^\beta}\frac{\partial q^\nu}{\partial Q^\gamma}g^{\kappa\lambda}\left(\frac{\partial g_{\lambda\nu}}{\partial q^\mu}+\frac{\partial g_{\lambda\mu}}{\partial q^\nu}-\frac{\partial g_{\mu\nu}}{\partial q^\lambda}\right)+\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial^2 q^\nu}{\partial Q^\beta \partial Q^\gamma}\underbrace{g^{\kappa\lambda}g_{\lambda\nu}}_{\delta^\kappa_\nu}
$$

$$
=\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\mu}{\partial Q^\beta}\frac{\partial q^\nu}{\partial Q^\gamma}\Gamma^\kappa_{\mu\nu}+\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial^2 q^\kappa}{\partial Q^\beta \partial Q^\gamma}.
$$

（証明終わり）

# 補遺：共変微分の共変性

共変微分 $\nabla_Y X$ が反変ベクトルの変換則に従うこと．

$X^\alpha$ を反変ベクトルとする．座標変換 $(q^\alpha) \to (Q^\alpha)$ で $(X^\alpha) \to (\widetilde{X}^\alpha)$ と変換するとして，$\nabla_\beta X^\alpha$ が

$$\nabla_\alpha X^\alpha \to \widetilde{\nabla}_\beta \widetilde{X}^\alpha = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\nabla_\lambda X^\kappa$$

と変換することを示せばよい．

$$\widetilde{\nabla}_\beta \widetilde{X}^\alpha = \frac{\partial \widetilde{X}^\alpha}{\partial Q^\beta} + \widetilde{\Gamma}^\alpha_{\beta\tau}\widetilde{X}^\tau$$

($\widetilde{\Gamma}^\alpha_{\beta\tau}$ は変換後の Christoffel の記号)

$$= \frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial}{\partial q^\lambda}\left(\frac{\partial Q^\alpha}{\partial q^\kappa}X^\kappa\right) + \widetilde{\Gamma}^\alpha_{\beta\tau}\frac{\partial Q^\tau}{\partial q^\mu}X^\mu$$

$$= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial X^\kappa}{\partial q^\lambda} + \frac{\partial^2 Q^\alpha}{\partial q^\lambda \partial q^\mu}\frac{\partial q^\lambda}{\partial Q^\beta}X^\mu + \widetilde{\Gamma}^\alpha_{\beta\tau}\frac{\partial Q^\tau}{\partial q^\mu}X^\mu$$

$$= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial X^\kappa}{\partial q^\lambda} + \underbrace{\delta^\alpha_\rho}_{\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\kappa}{\partial Q^\gamma}}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial^2 Q^\gamma}{\partial q^\lambda \partial q^\mu}X^\mu$$

# 補遺：共変微分の共変性

$$
\begin{aligned}
&+ \underbrace{\delta^{\alpha}_{\rho}}_{\frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\kappa}{\partial Q^\rho}} \underbrace{\delta^{\sigma}_{\beta}}_{\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial Q^\sigma}{\partial q^\lambda}} \widetilde{\Gamma}^{\rho}_{\sigma\tau}\frac{\partial Q^\tau}{\partial q^\mu}X^\mu \\
&= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta} \\
&= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\left\{\frac{\partial X^\kappa}{\partial q^\lambda} + \left(\frac{\partial q^\kappa}{\partial Q^\rho}\frac{\partial Q^\sigma}{\partial q^\lambda}\frac{\partial Q^\tau}{\partial q^\mu}\widetilde{\Gamma}^{\rho}_{\sigma\tau} + \frac{\partial q^\kappa}{\partial Q^\gamma}\frac{\partial^2 Q^\gamma}{\partial q^\lambda \partial q^\mu}\right)X^\mu\right\}.
\end{aligned}
$$

Christoffel の記号の変換則により青字部分は $\Gamma^{\kappa}_{\lambda\mu}$ に等しいから，

$$
\widetilde{\nabla}_\beta \widetilde{X}^\alpha = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\left(\frac{\partial X^\kappa}{\partial q^\lambda} + \Gamma^{\kappa}_{\lambda\mu}X^\mu\right) = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\nabla_\lambda X^\kappa.
$$

（証明終わり）

# 補遺：曲率の表式の証明

まず，曲率 $R_{\kappa\lambda\mu\nu}$ は次のように表される．

$$R_{\kappa\lambda\mu\nu} = g_{\kappa\rho}(\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}).$$

$$\begin{aligned}
& g_{\kappa\rho}(\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda}) \\
&= \frac{1}{2} g_{\kappa\rho} \left\{ \partial_\mu \left[ g^{\rho\sigma}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}) \right] - \partial_\nu \left[ g^{\rho\sigma}(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) \right] \right\} \\
&= \frac{1}{2} \Big\{ \underbrace{g_{\kappa\rho} g^{\rho\sigma}}_{\delta^\sigma_\kappa} \partial_\mu(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}) + g_{\kappa\rho}\partial_\mu g^{\rho\sigma}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}) \\
&\quad - \underbrace{g_{\kappa\rho} g^{\rho\sigma}}_{\delta^\sigma_\kappa} \partial_\nu(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) - g_{\kappa\rho}\partial_\nu g^{\rho\sigma}(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) \Big\}
\end{aligned}$$

（$g_{\kappa\rho} g^{\rho\sigma} = \delta^\sigma_\kappa$ の両辺を微分して $g_{\kappa\rho}\partial_\mu g^{\rho\sigma} = -g^{\rho\sigma}\partial_\mu g_{\kappa\rho}$ を得るから）

$$\begin{aligned}
&= \frac{1}{2} \Big\{ \partial_\mu(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\kappa\nu} - \partial_\kappa g_{\nu\lambda}) - g^{\rho\sigma}\partial_\mu g_{\kappa\rho}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\lambda\nu}) \\
&\quad - \partial_\nu(\partial_\mu g_{\kappa\lambda} + \partial_\lambda g_{\kappa\mu} - \partial_\kappa g_{\mu\lambda}) + g^{\rho\sigma}\partial_\nu g_{\kappa\rho}(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}) \Big\} \\
&= \frac{1}{2}(\partial_\lambda\partial_\mu g_{\kappa\nu} + \partial_\kappa\partial_\nu g_{\lambda\mu} - \partial_\kappa\partial_\mu g_{\lambda\nu} - \partial_\lambda\partial_\nu g_{\kappa\mu}) - \partial_\mu g_{\kappa\rho}\Gamma^\rho_{\lambda\nu} + \partial_\nu g_{\kappa\rho}\Gamma^\rho_{\lambda\mu}.
\end{aligned}$$

# 補遺：曲率の表式の証明

$$
\begin{aligned}
& -\partial_\mu g_{\kappa\rho}\Gamma^\rho_{\lambda\nu} + g_{\kappa\rho}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} \\
&= (-\partial_\mu g_{\kappa\sigma} + g_{\kappa\rho}\Gamma^\rho_{\mu\sigma})\Gamma^\sigma_{\nu\lambda} \\
&= \left[-\partial_\mu g_{\kappa\sigma} + \frac{1}{2}(\partial_\mu g_{\kappa\sigma} + \partial_\sigma g_{\kappa\mu} - \partial_\kappa g_{\mu\sigma})\right]\Gamma^\sigma_{\nu\lambda} \\
&= -\frac{1}{2}(\partial_\kappa g_{\mu\sigma} + \partial_\mu g_{\kappa\sigma} - \partial_\sigma g_{\kappa\mu})\Gamma^\sigma_{\nu\lambda} \\
&= -g_{\sigma\tau}\Gamma^\tau_{\kappa\mu}\Gamma^\sigma_{\lambda\nu}.
\end{aligned}
$$

同様にして，

$$
\partial_\nu g_{\kappa\rho}\Gamma^\rho_{\lambda\mu} - g_{\kappa\rho}\Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} = g_{\sigma\tau}\Gamma^\tau_{\kappa\nu}\Gamma^\sigma_{\lambda\mu}
$$

を得る．以上により題意の等式を得る．

（証明終わり）