$\mathbb{R}^3$ の場合、一般座標 (q^1, q^2, q^3)

2形式 η に対し $**\eta = \eta$ を確かめる。

$$\eta = \eta_{23}\, dq^2 \wedge dq^3 + \eta_{31}\, dq^3 \wedge dq^1 + \eta_{12}\, dq^1 \wedge dq^2 \quad \text{とおく.}$$

$$**\eta = \sqrt{g}\,(\omega^1 dq^2 \wedge dq^3 + \omega^2 dq^3 \wedge dq^1 + \omega^3 dq^1 \wedge dq^2),$$

ここで

$$\omega_1 = \sqrt{g}\,\eta^{23}, \quad \omega_2 = \sqrt{g}\,\eta^{31}, \quad \omega_3 = \sqrt{g}\,\eta^{12}.$$

$$\begin{aligned}\omega^1 &= g^{1\lambda}\omega_\lambda = g^{11}\omega_1 + g^{12}\omega_2 + g^{13}\omega_3 \\ &= \sqrt{g}\,(g^{11}\eta^{23} + g^{12}\eta^{31} + g^{13}\eta^{12}), \qquad \text{——①}\end{aligned}$$

$$\omega^2 = g^{2\lambda}\omega_\lambda, \quad \omega^3 = g^{3\lambda}\omega_\lambda.$$

$$\begin{aligned}\eta^{23} &= g^{2\mu} g^{3\nu} \eta_{\mu\nu} \\ &= g^{22} g^{33} \eta_{23} + g^{23} g^{32} \underbrace{\eta_{32}}_{-\eta_{23}} + g^{23} g^{31} \eta_{31} + g^{21} g^{33} \underbrace{\eta_{13}}_{-\eta_{31}} \\ &\quad + g^{21} g^{32} \eta_{12} + g^{22} g^{3} \underbrace{\eta_{21}}_{-\eta_{12}} \\ &= \begin{vmatrix} g^{22} & g^{23} \\ g^{32} & g^{33} \end{vmatrix} \eta_{23} - \begin{vmatrix} g^{21} & g^{23} \\ g^{31} & g^{33} \end{vmatrix} \eta_{31} + \begin{vmatrix} g^{21} & g^{22} \\ g^{31} & g^{32} \end{vmatrix} \eta_{12},\end{aligned}$$

同様にして、

$$\eta^{31} = - \begin{vmatrix} g^{12} & g^{13} \\ g^{32} & g^{33} \end{vmatrix} \eta_{23} + \begin{vmatrix} g^{11} & g^{13} \\ g^{31} & g^{33} \end{vmatrix} \eta_{31} - \begin{vmatrix} g^{11} & g^{12} \\ g^{31} & g^{32} \end{vmatrix} \eta_{12},$$

$$\eta^{12} = \begin{vmatrix} g^{12} & g^{13} \\ g^{22} & g^{23} \end{vmatrix} \eta_{23} - \begin{vmatrix} g^{11} & g^{13} \\ g^{21} & g^{23} \end{vmatrix} \eta_{31} + \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix} \eta_{12}.$$

これらを①に代入して、

$$\begin{aligned}\omega^1 = \sqrt{g}\Bigg\{ &\underbrace{\left(g^{11} \begin{vmatrix} g^{22} & g^{23} \\ g^{32} & g^{33} \end{vmatrix} - g^{21} \begin{vmatrix} g^{12} & g^{13} \\ g^{32} & g^{33} \end{vmatrix} + g^{31} \begin{vmatrix} g^{12} & g^{13} \\ g^{22} & g^{23} \end{vmatrix} \right)}_{\det(g^{\mu\nu}) = g^{-1}} \eta_{23} \\ &- \underbrace{\left(g^{11} \begin{vmatrix} g^{21} & g^{23} \\ g^{31} & g^{33} \end{vmatrix} - g^{21} \begin{vmatrix} g^{11} & g^{13} \\ g^{31} & g^{33} \end{vmatrix} + g^{31} \begin{vmatrix} g^{11} & g^{13} \\ g^{21} & g^{23} \end{vmatrix} \right)}_{0} \eta_{31} \\ &+ \underbrace{\left(g^{11} \begin{vmatrix} g^{21} & g^{22} \\ g^{31} & g^{32} \end{vmatrix} - g^{21} \begin{vmatrix} g^{11} & g^{12} \\ g^{31} & g^{32} \end{vmatrix} + g^{31} \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix} \right)}_{0} \eta_{12} \\ = g^{-1/2} \eta_{23}.&\end{aligned}$$

同様に $\omega^2 = g^{-1/2}\eta_{31}$, $\omega^3 = g^{-1/2}\eta_{12}$ を得る。ゆえに、$**\eta = \eta$. ⊠

$**\omega = (-1)^{p(N-p)}(\mathrm{sgn}\, g)\,\omega$ （ω : p形式）の証明

$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\, d\xi^{\mu_1}\wedge\cdots\wedge d\xi^{\mu_p}$ とおく.

$$
\begin{aligned}
**\omega &= \frac{1}{(p!)^2(N-p)!} E^{\nu_{p+1}\cdots\nu_N}{}_{\rho_1\cdots\rho_p} E^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_N}\,\omega_{\mu_1\cdots\mu_p}\, d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p} \\
&= \frac{(-1)^{p(N-p)}}{(p!)^2(N-p)!} E_{\rho_1\cdots\rho_p\nu_{p+1}\cdots\nu_N} E^{\mu_1\cdots\mu_p\nu_{p+1}\cdots\nu_N}\,\omega_{\mu_1\cdots\mu_p}\, d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p} \\
&= \frac{(-1)^{p(N-p)}}{(p!)^2(N-p)!}(\mathrm{sgn}\, g)(N-p)! \sum_{\sigma\in\mathfrak{S}_p}(\mathrm{sgn}\,\sigma)\,\delta^{\mu_1}_{\rho_{\sigma(1)}}\cdots\delta^{\mu_p}_{\rho_{\sigma(p)}}\,\omega_{\mu_1\cdots\mu_p} \\
&\qquad \times d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p}
\end{aligned}
$$

[$\mathfrak{S}_p$: $(1\ 2\cdots p)$ の置換全体の集合]

$$
\begin{aligned}
&= \frac{(-1)^{p(N-p)}}{(p!)^2}(\mathrm{sgn}\, g)\sum_{\sigma\in\mathfrak{S}_p}(\mathrm{sgn}\,\sigma)\,\omega_{\rho_{\sigma(1)}\cdots\rho_{\sigma(p)}}\, d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p} \\
&= \frac{(-1)^{p(N-p)}}{(p!)^2}(\mathrm{sgn}\, g)\sum_{\sigma\in\mathfrak{S}_p}\omega_{\rho_1\cdots\rho_p}\, d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p} \\
&= \frac{(-1)^{p(N-p)}}{p!}(\mathrm{sgn}\, g)\,\omega_{\rho_1\cdots\rho_p}\, d\xi^{\rho_1}\wedge\cdots\wedge d\xi^{\rho_p} = (-1)^{p(N-p)}(\mathrm{sgn}\, g)\,\omega .
\end{aligned}
$$

⊠

[注意]

$$
\begin{aligned}
E^{\mu_1\cdots\mu_N} &:= g^{\mu_1\nu_1}\cdots g^{\mu_N\nu_N}E_{\nu_1\cdots\nu_N} \\
&= \frac{\mathrm{sgn}\, g}{\sqrt{|g|}} \times \begin{cases} +1 & (1\ 2\cdots N)\to(\mu_1\cdots\mu_N) \text{ は偶置換} \\ -1 & (1\ 2\cdots N)\to(\mu_1\cdots\mu_N) \text{ は奇置換} \\ 0 & \text{その他} \end{cases}
\end{aligned}
$$

$$
E_{\lambda_1\cdots\lambda_p\nu_{p+1}\cdots\nu_N}E^{\mu_1\cdots\mu_p\nu_{p+1}\cdots\nu_N} = (\mathrm{sgn}\, g)(N-p)!\sum_{\sigma\in\mathfrak{S}_p}(\mathrm{sgn}\,\sigma)\,\delta^{\mu_1}_{\lambda_{\sigma(1)}}\cdots\delta^{\mu_p}_{\lambda_{\sigma(p)}} .
$$

$\omega \wedge * \psi$ の表式の証明

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\, dq^{\mu_1}\wedge\cdots\wedge dq^{\mu_p}, \quad \psi = \frac{1}{p!}\psi_{\mu_1\cdots\mu_p}\, dq^{\mu_1}\wedge\cdots\wedge dq^{\mu_p}$$

のとき、

$$\omega \wedge * \psi = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\psi^{\mu_1\cdots\mu_p}\,\Omega_{\mathrm{vol}}.$$

$\because$

$$\omega\wedge *\psi = \frac{1}{(p!)^2(N-p)!} E^{\nu_1\cdots\nu_p}{}_{\nu_{p+1}\cdots\nu_N}\,\omega_{\mu_1\cdots\mu_p}\psi_{\nu_1\cdots\nu_p}\, dq^{\mu_1}\wedge\cdots\wedge dq^{\mu_p} \wedge dq^{\nu_{p+1}}\wedge\cdots\wedge dq^{\nu_N}$$

$$= \frac{1}{(p!)^2(N-p)!} \boxed{E_{\nu_1\cdots\nu_p\nu_{p+1}\cdots\nu_N}}\,\omega_{\mu_1\cdots\mu_p}\psi^{\nu_1\cdots\nu_p} \times (\mathrm{sgn}\, g)\sqrt{|g|}\,\boxed{E^{\mu_1\cdots\mu_p\nu_{p+1}\cdots\nu_N}}\, dq^1\wedge\cdots\wedge dq^N$$

$$= \frac{1}{(p!)^2(N-p)!}\boxed{(\mathrm{sgn}\, g)^2 (N-p)! \sum_{\sigma\in\mathfrak{S}_p}(\mathrm{sgn}\,\sigma)\,\delta^{\mu_1}_{\nu_{\sigma(1)}}\cdots\delta^{\mu_p}_{\nu_{\sigma(p)}}}\,\omega_{\mu_1\cdots\mu_p}\psi^{\nu_1\cdots\nu_p}\,\Omega_{\mathrm{vol}}$$

$$= \frac{1}{(p!)^2}\sum_{\sigma\in\mathfrak{S}_p}(\mathrm{sgn}\,\sigma)\,\omega_{\nu_{\sigma(1)}\cdots\nu_{\sigma(p)}}\psi^{\nu_1\cdots\nu_p}\,\Omega_{\mathrm{vol}}$$

$$= \frac{1}{(p!)^2}\sum_{\sigma\in\mathfrak{S}_p}\omega_{\nu_1\cdots\nu_p}\psi^{\nu_1\cdots\nu_p}\,\Omega_{\mathrm{vol}} = \frac{1}{p!}\omega_{\nu_1\cdots\nu_p}\psi^{\nu_1\cdots\nu_p}\,\Omega_{\mathrm{vol}}. \qquad \boxtimes$$

余微分の表式

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

のとき,

$$d^\dagger \omega = -\frac{1}{(p-1)!}\nabla_\lambda \omega^\lambda{}_{\mu_1\cdots\mu_{p-1}}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p-1}}.$$

$$\because\quad d*\omega = \frac{1}{p!(N-p)!}\partial_\lambda\left(E^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_N}\,\omega_{\mu_1\cdots\mu_p}\right) dx^\lambda\wedge dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_N}$$

$$= \frac{1}{p!(N-p)!}\partial_\lambda\left(E_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_N}\,\omega^{\mu_1\cdots\mu_p}\right) dx^\lambda\wedge dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_N}$$

$$= \frac{1}{p!(N-p)!}\frac{1}{\sqrt{|g|}}E_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_N}\,\partial_\lambda\left(\sqrt{|g|}\,\omega^{\mu_1\cdots\mu_p}\right) dx^\lambda\wedge dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_N},$$

$$*d*\omega = \frac{1}{p!(p-1)!(N-p)!}\frac{1}{\sqrt{|g|}}E^{\lambda\mu_{p+1}\cdots\mu_N}{}_{\nu_1\cdots\nu_{p-1}}E_{\mu_1\cdots\mu_N}\,\partial_\lambda\left(\sqrt{|g|}\,\omega^{\mu_1\cdots\mu_p}\right)$$
$$\times dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{p!(p-1)!(N-p)!}\frac{1}{\sqrt{|g|}}g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}E^{\lambda\rho_1\cdots\rho_{p-1}\mu_{p+1}\cdots\mu_N}E_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_N}$$
$$\times\partial_\lambda\left(\sqrt{|g|}\,\omega^{\mu_1\cdots\mu_p}\right) dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{p!(p-1)!(N-p)!}\frac{1}{\sqrt{|g|}}g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}(\operatorname{sgn} g)\sum_{\sigma\in\mathfrak{S}_p}(\operatorname{sgn}\sigma)\,\delta^\lambda_{\mu_{\sigma(1)}}\delta^{\rho_1}_{\mu_{\sigma(2)}}\cdots\delta^{\rho_{p-1}}_{\mu_{\sigma(p)}}$$
$$\times\partial_\lambda\left(\sqrt{|g|}\,\omega^{\mu_1\cdots\mu_p}\right) dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{p!(p-1)!}\frac{1}{\sqrt{|g|}}(\operatorname{sgn} g)\,g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}\sum_{\sigma\in\mathfrak{S}_p}\delta^\lambda_{\mu_{\sigma(1)}}\delta^{\rho_1}_{\mu_{\sigma(2)}}\cdots\delta^{\rho_{p-1}}_{\mu_{\sigma(p)}}$$
$$\times\partial_\lambda\left(\sqrt{|g|}\,\omega^{\mu_{\sigma(1)}\cdots\mu_{\sigma(p)}}\right) dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{p!(p-1)!}\frac{1}{\sqrt{|g|}}(\operatorname{sgn} g)\,g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}\sum_{\sigma\in\mathfrak{S}_p}\partial_\lambda\left(\sqrt{|g|}\,\omega^{\lambda\rho_1\cdots\rho_{p-1}}\right)$$
$$\times dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{(p-1)!}\frac{1}{\sqrt{|g|}}(\operatorname{sgn} g)\,g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}\,\partial_\lambda\left(\sqrt{|g|}\,\omega^{\lambda\rho_1\cdots\rho_{p-1}}\right) dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p-1}}.$$

ここで

$$\nabla_\lambda\omega^{\lambda\nu_1\cdots\nu_{p-1}} = \partial_\lambda\omega^{\lambda\rho_1\cdots\rho_{p-1}} + \Gamma^\lambda_{\lambda\kappa}\omega^{\kappa\rho_1\cdots\rho_{p-1}}$$
$$\underbrace{+\underbrace{\Gamma^{\rho_1}_{\lambda\kappa}}_{\lambda,\kappa\text{について対称}}\underbrace{\omega^{\lambda\kappa\rho_2\cdots\rho_{p-1}}}_{\lambda,\kappa\text{について反対称}} + \cdots + \Gamma^{\rho_{p-1}}_{\lambda\kappa}\omega^{\lambda\rho_1\cdots\rho_{p-2}\kappa}}_{0}$$

$$= \partial_\lambda\omega^{\lambda\rho_1\cdots\rho_{p-1}} + \Gamma^\lambda_{\lambda\kappa}\omega^{\kappa\rho_1\cdots\rho_{p-1}}.$$

NO.

DATE

$$\Gamma^{\lambda}_{\mu\kappa} = \frac{1}{2} g^{\lambda\nu}(\partial_\mu g_{\nu\kappa} + \partial_\kappa g_{\nu\mu} - \partial_\nu g_{\mu\kappa}),$$

$$\Gamma^{\lambda}_{\lambda\kappa} = \frac{1}{2} g^{\lambda\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\nu\lambda} - \partial_\nu g_{\lambda\kappa}) = \frac{1}{2} g^{\lambda\nu}\partial_\kappa g_{\lambda\nu}.$$

$(g^{\mu\nu})$ は $(g_{\mu\nu})$ の逆行列であることから、

$$g^{\lambda\nu} = (-1)^{\lambda+\nu} g^{-1} \tilde{g}^{\nu\lambda},$$

$\tilde{g}^{\nu\lambda} :=$ [$|g_{\alpha\beta}|$ から ν行λ列を除いた小行列式]

であるので、

$$\begin{aligned}
\partial_\kappa g &= \partial_\kappa \det(g_{\alpha\beta}) \\
&= \begin{vmatrix} \partial_\kappa g_{11} & \cdots & \partial_\kappa g_{1N} \\ g_{21} & \cdots & g_{2N} \\ \vdots & & \vdots \\ g_{N1} & \cdots & g_{NN} \end{vmatrix} + \cdots + \begin{vmatrix} g_{11} & \cdots & g_{1N} \\ \vdots & & \vdots \\ \partial_\kappa g_{N1} & \cdots & \partial_\kappa g_{NN} \end{vmatrix} \\
&= \sum_{\lambda,\nu} (-1)^{\lambda+\nu} \tilde{g}^{\nu\lambda} \partial_\kappa g_{\nu\lambda} \\
&= g\, g^{\lambda\nu} \partial_\kappa g_{\lambda\nu},
\end{aligned}$$

$$\Gamma^{\lambda}_{\lambda\kappa} = \frac{1}{2} g^{\lambda\nu}\partial_\kappa g_{\lambda\nu} = \frac{1}{2g}\partial_\kappa g = \frac{1}{\sqrt{|g|}}\partial_\kappa \sqrt{|g|}.$$

よって、

$$\begin{aligned}
\nabla_\lambda \omega^{\lambda\rho_1\cdots\rho_{p-1}} &= \partial_\lambda \omega^{\lambda\rho_1\cdots\rho_{p-1}} + \frac{1}{\sqrt{|g|}}\partial_\lambda\sqrt{|g|}\cdot\omega^{\lambda\rho_1\cdots\rho_{p-1}} \\
&= \frac{1}{\sqrt{|g|}}\partial_\lambda\left(\sqrt{|g|}\,\omega^{\lambda\rho_1\cdots\rho_{p-1}}\right),
\end{aligned}$$

$$*d*\omega = \frac{(-1)^{(p-1)(N-p)}}{(p-1)!}(\mathrm{sgn}\, g)\, g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}} \nabla_\lambda \omega^{\lambda\rho_1\cdots\rho_{p-1}}\, dq^{\nu_1}\wedge\cdots\wedge dq^{\nu_{p-1}}$$

($\nabla_\lambda g_{\mu\nu} = 0$ であるから)

$$= \frac{(-1)^{(p-1)(N-p)}}{(p-1)!}(\mathrm{sgn}\, g)\, \nabla_\lambda\left(g_{\nu_1\rho_1}\cdots g_{\nu_{p-1}\rho_{p-1}}\omega^{\lambda\rho_1\cdots\rho_{p-1}}\right) dq^{\nu_1}\wedge\cdots\wedge dq^{\nu_{p-1}}$$

$$= \frac{(-1)^{(p-1)(N-p)}}{(p-1)!}(\mathrm{sgn}\, g)\, \nabla_\lambda \omega^{\lambda}{}_{\nu_1\cdots\nu_{p-1}}\, dq^{\nu_1}\wedge\cdots\wedge dq^{\nu_{p-1}}.$$

$$\therefore\ d^\dagger\omega = (-1)^{Np+N+1}(\mathrm{sgn}\, g) *d*\omega = -\frac{1}{(p-1)!}\nabla_\lambda\omega^{\lambda}{}_{\nu_1\cdots\nu_{p-1}}\, dq^{\nu_1}\wedge\cdots\wedge dq^{\nu_{p-1}}.$$

⊠