$\delta\int d^4x\,\sqrt{-g}\,\mathcal{L}_{gravity}$ の計算

$R = g^{\mu\nu}R_{\mu\nu}$ と

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu} \qquad ①$$

を用いて、

$$\delta\int d^4x\,\sqrt{-g}\,\mathcal{L}_{gravity} = \frac{1}{2c\kappa}\int d^4x\,\delta\left(\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}\right)$$

$$= \frac{1}{2c\kappa}\int d^4x\left[\delta(\sqrt{-g})\,g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\right]$$

$$= \frac{1}{2c\kappa}\int d^4x\left[-\frac{1}{2}\sqrt{-g}\,g_{\rho\sigma}\,\delta g^{\rho\sigma}R + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}\right]$$

$$= \frac{1}{2c\kappa}\int d^4x\,\sqrt{-g}\left[\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + g^{\mu\nu}\delta R_{\mu\nu}\right].$$

ここで次が成り立つ：

$$\delta R_{\mu\nu} = \nabla_\rho\,\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\,\delta R^\rho_{\mu\rho}.$$

(∵ $R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\gamma\rho}\Gamma^\gamma_{\mu\nu} - \Gamma^\rho_{\gamma\nu}\Gamma^\gamma_{\mu\rho}$,

$$\delta R_{\mu\nu} = \underbrace{\partial_\rho(\delta\Gamma^\rho_{\mu\nu})}_{(1)} \underbrace{- \partial_\nu(\delta\Gamma^\rho_{\rho\mu})}_{(2)} \underbrace{+ \delta\Gamma^\rho_{\gamma\rho}\Gamma^\gamma_{\mu\nu}}_{(3)} \underbrace{+ \Gamma^\rho_{\gamma\rho}\delta\Gamma^\gamma_{\mu\nu}}_{(4)}$$
$$\underbrace{- \delta\Gamma^\rho_{\gamma\nu}\Gamma^\gamma_{\mu\rho}}_{(5)} \underbrace{- \Gamma^\rho_{\gamma\nu}\delta\Gamma^\gamma_{\mu\rho}}_{(6)}.$$

ここで、

$\delta\Gamma^\rho_{\mu\nu}$ は(1, 2)-型テンソルである ②

ことに注意すれば、

$$\nabla_\rho(\delta\Gamma^\rho_{\mu\nu}) = \underbrace{\partial_\rho(\delta\Gamma^\rho_{\mu\nu})}_{(1)} \underbrace{+ \Gamma^\rho_{\rho\gamma}\delta\Gamma^\gamma_{\mu\nu}}_{(4)} \underbrace{- \Gamma^\gamma_{\rho\mu}\delta\Gamma^\rho_{\gamma\nu}}_{(5)} - \Gamma^\gamma_{\rho\nu}\delta\Gamma^\rho_{\mu\gamma},$$

$$\nabla_\nu(\delta\Gamma^\rho_{\rho\mu}) = \underbrace{\partial_\nu(\delta\Gamma^\rho_{\rho\mu})}_{-(2)} \underbrace{+ \Gamma^\rho_{\nu\gamma}\delta\Gamma^\gamma_{\rho\mu}}_{-(6)} - \Gamma^\gamma_{\nu\rho}\delta\Gamma^\rho_{\gamma\mu} \underbrace{- \Gamma^\gamma_{\nu\mu}\delta\Gamma^\rho_{\rho\gamma}}_{+(3)}$$

であることがわかるので、

$$\delta R_{\mu\nu} = \nabla_\rho(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu}) \cancel{+ \Gamma^\gamma_{\rho\nu}\delta\Gamma^\rho_{\mu\gamma}} \cancel{- \Gamma^\gamma_{\nu\rho}\delta\Gamma^\rho_{\gamma\mu}}$$
$$= \nabla_\rho(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu}).$$

⊠)

したがって、

$$\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}\left[\nabla_\rho(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\mu})\right]$$
$$= \sqrt{-g}\left[\nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\rho_{\rho\mu})\right]$$

(計量テンソルの共変微分はゼロであるから)

（一般に，反変ベクトル A^ρ に対し，

$$\nabla_\rho A^\rho = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}A^\rho) \quad \text{——③}$$

が成り立つから）

$$= \partial_\rho\left[\sqrt{-g}\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\alpha_{\alpha\mu}\right)\right]$$

となり

$$\delta\int d^4x\sqrt{-g}\,\mathcal{L}_{gravity}$$

$$= \frac{1}{2c\kappa}\int d^4x\sqrt{-g}\;\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

$$+ \frac{1}{2c\kappa}\underbrace{\int d^4x\;\partial_\rho\left[\sqrt{-g}\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\alpha_{\alpha\mu}\right)\right]}_{\text{境界積分}=0}$$

$$= \frac{1}{2c\kappa}\int d^4x\;\delta g^{\mu\nu}\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right).$$

$$\therefore\ \boxed{\begin{aligned}&\delta\int d^4x\sqrt{-g}\,\mathcal{L}_{gravity} = \frac{1}{2c\kappa}\int d^4x\;\delta g^{\mu\nu}\sqrt{-g}\,G_{\mu\nu},\\ &G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\quad(\text{Einstein テンソル}).\end{aligned}}$$

①，②，③ の証明が残されている．

① $\delta g = g g^{\mu\nu}\delta g_{\mu\nu}$ が成り立つ（$\partial_\lambda g = g g^{\mu\nu}\partial_\lambda g_{\mu\nu}$ と同様にして示される）．
$(g_{\mu\nu}) \to (g^{\mu\nu})$（逆行列）とすることにより $\delta(g^{-1}) = g^{-1}g_{\mu\nu}\delta g^{\mu\nu}$．
$\delta(g^{-1}) = -g^{-2}\delta g$ を代入して，$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$ を得る．これより

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g g_{\mu\nu},\quad \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\frac{\partial g}{\partial g^{\mu\nu}} = \frac{1}{2}\sqrt{-g}\,g_{\mu\nu}. \quad ⊠$$

② $\Gamma^\kappa_{\mu\nu}$ を計量 $g_{\mu\nu}$ から定まる Christoffel の記号，$\tilde{\Gamma}^\kappa_{\mu\nu}$ を計量 $g_{\mu\nu}+\delta g_{\mu\nu}$ から定まる Christoffel の記号とする．座標変換 $x^\mu \to y^\mu$ により両者はそれぞれ

$$\Gamma^\kappa_{\mu\nu} \to \Gamma'^\kappa_{\mu\nu} = \frac{\partial y^\kappa}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial y^\kappa}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu\partial y^\nu},$$

$$\tilde{\Gamma}^\kappa_{\mu\nu} \to \tilde{\Gamma}'^\kappa_{\mu\nu} = \frac{\partial y^\kappa}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}\tilde{\Gamma}^\alpha_{\beta\gamma} + \frac{\partial y^\kappa}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu\partial y^\nu}$$

と変換する．差をとると，$\delta\Gamma^\kappa_{\mu\nu} := \tilde{\Gamma}^\kappa_{\mu\nu} - \Gamma^\kappa_{\mu\nu}$ は

$$\delta\Gamma^\kappa_{\mu\nu} \to \delta\Gamma'^\kappa_{\mu\nu} = \frac{\partial y^\kappa}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}\delta\Gamma^\alpha_{\beta\gamma}$$

と変換することがわかるので，$\delta\Gamma^\kappa_{\mu\nu}$ は (1,2)-型テンソルである． ⊠

③ 反変ベクトル A^μ に対し、

$$\nabla_\rho A^\rho = \partial_\rho A^\rho + \Gamma^\rho_{\rho\sigma} A^\sigma$$

となる。そこで、$\Gamma^\rho_{\rho\sigma}$ を求める。

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

より

$$\Gamma^\rho_{\rho\sigma} = \frac{1}{2} g^{\rho\tau}(\cancel{\partial_\rho g_{\tau\sigma}} + \partial_\sigma g_{\tau\rho} \cancel{- \partial_\tau g_{\rho\sigma}}) = \frac{1}{2} g^{\rho\tau}\partial_\sigma g_{\rho\tau}.$$

$$= \frac{1}{2}\frac{1}{g}\partial_\sigma g = \frac{1}{\sqrt{-g}}\partial_\sigma \sqrt{-g}.$$

$$\therefore \nabla_\rho A^\rho = \partial_\rho A^\rho + \frac{1}{\sqrt{-g}}\partial_\sigma\sqrt{-g}\cdot A^\sigma = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}A^\rho).$$

⊠