運動曲面の接ベクトルしか眼中にない
束縛力(⊥運動曲面)は見ない



$\tilde{A}^\alpha = \frac{\partial Q^\alpha}{\partial q^\kappa} A^\kappa$ を示すには，次のChristoffel記号の変換則を用いる：

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma}\Gamma^\kappa_{\lambda\mu} + \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{\partial^2 q^\kappa}{\partial Q^\beta \partial Q^\gamma}. \quad \text{— ⓓ}$$

$$\frac{dQ^\alpha}{dt} = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{dq^\kappa}{dt}$$ （一般速度 $\left(\frac{dq^\alpha}{dt}\right)$ は反変ベクトル場の成分である）．

$$\frac{d^2Q^\alpha}{dt^2} = \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{d^2q^\kappa}{dt^2} + \frac{\partial^2 Q^\alpha}{\partial q^\kappa \partial q^\lambda}\frac{dq^\kappa}{dt}\frac{dq^\lambda}{dt}.$$

$$\tilde{A}^\alpha = \frac{d^2Q^\alpha}{dt^2} + \tilde{\Gamma}^\alpha_{\beta\gamma}\frac{dQ^\beta}{dt}\frac{dQ^\gamma}{dt}$$

$$= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{d^2q^\kappa}{dt^2} + \frac{\partial^2 Q^\alpha}{\partial q^\kappa \partial q^\lambda}\frac{dq^\kappa}{dt}\frac{dq^\lambda}{dt} + \tilde{\Gamma}^\alpha_{\beta\gamma}\frac{\partial Q^\beta}{\partial q^\kappa}\frac{dq^\kappa}{dt}\frac{\partial Q^\gamma}{\partial q^\lambda}\frac{dq^\lambda}{dt}$$

$$= \frac{\partial Q^\alpha}{\partial q^\kappa}\frac{d^2q^\kappa}{dt^2} + \underbrace{\left(\frac{\partial Q^\beta}{\partial q^\kappa}\frac{\partial Q^\gamma}{\partial q^\lambda}\tilde{\Gamma}^\alpha_{\beta\gamma} + \frac{\partial^2 Q^\alpha}{\partial q^\kappa \partial q^\lambda}\right)}\frac{dq^\kappa}{dt}\frac{dq^\lambda}{dt}$$

$$\delta^\alpha_\delta\left(\frac{\partial Q^\beta}{\partial q^\kappa}\frac{\partial Q^\gamma}{\partial q^\lambda}\tilde{\Gamma}^\delta_{\beta\gamma} + \frac{\partial^2 Q^\delta}{\partial q^\kappa \partial q^\lambda}\right)$$

$$= \frac{\partial Q^\alpha}{\partial q^\mu}\frac{\partial q^\mu}{\partial Q^\delta}\left(\cdots\right)$$

$$= \frac{\partial Q^\alpha}{\partial q^\mu}\Gamma^\mu_{\kappa\lambda}$$ ←［ⓓで$(q^\alpha)$と$(Q^\alpha)$の役割を入れ換えた式より］

$$= \frac{\partial Q^\alpha}{\partial q^\kappa}\left(\frac{d^2q^\kappa}{dt^2} + \Gamma^\kappa_{\lambda\mu}\frac{dq^\lambda}{dt}\frac{dq^\mu}{dt}\right) = \frac{\partial Q^\alpha}{\partial q^\kappa}A^\kappa.$$

~~ⓓの証明：~~

$$\cancel{\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}\tilde{g}^{\alpha\delta}\left(\frac{\partial \tilde{g}_{\delta\beta}}{\partial Q^\gamma} + \frac{\partial \tilde{g}_{\delta\gamma}}{\partial Q^\beta} - \frac{\partial \tilde{g}_{\beta\gamma}}{\partial Q^\delta}\right)}.$$

$$\cancel{\frac{\partial \tilde{g}_{\delta\beta}}{\partial Q^\gamma} = \frac{\partial}{\partial Q^\gamma}\left(\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial q^\lambda}{\partial Q^\beta}g_{\kappa\lambda}\right)} =$$

$$\cancel{= \frac{\partial g_{\kappa\lambda}}{\partial Q^\gamma}\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial q^\lambda}{\partial Q^\beta} + g_{\kappa\lambda}\left(\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial^2 q^\kappa}{\partial Q^\gamma \partial Q^\delta} + \frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}\right)}$$

~~これらを用いて，~~

$$\cancel{\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}\frac{\partial Q^\alpha}{\partial q^\mu}\frac{\partial Q^\delta}{\partial q^\nu}g^{\mu\nu}\left\{\frac{\partial g_{\kappa\lambda}}{\partial Q^\gamma}\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial q^\lambda}{\partial Q^\beta} + g_{\kappa\lambda}\left(\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial^2 q^\kappa}{\partial Q^\gamma \partial Q^\delta} + \frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}\right)\right.}$$

~~ⓓの証明：まず~~

$$\cancel{\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial q^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\right)}$$

~~の意味を求める，~~

「幾何学的運動方程式」は扱いにくいので，書き直す．

$$g_{\alpha\beta}\frac{d^2 q^\beta}{dt^2} + \Gamma_{\alpha\beta\gamma}\frac{dq^\beta}{dt}\frac{dq^\gamma}{dt} = f_\alpha, \qquad \text{—①}$$

$$\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial q^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\right),$$

$$f_\alpha := \sum_{i=1}^{N_p} \mathbb{F}_i \cdot \frac{\partial \mathbb{r}_i}{\partial q^\alpha}.$$

運動エネルギー $T := \sum_{i=1}^{N_p}\frac{m_i}{2}\|\dot{\mathbb{r}}_i\|^2 = \frac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$

$\parallel$

$T(q, \dot{q})$ $(q^\alpha)$と$(\dot{q}^\alpha)$は独立変数とみなす．

$$\text{①左辺} = g_{\alpha\beta}\ddot{q}^\beta + \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial q^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\right)\dot{q}^\beta\dot{q}^\gamma$$

$$= \frac{1}{2}\left(g_{\alpha\beta}\ddot{q}^\beta + g_{\alpha\gamma}\ddot{q}^\gamma\right) + \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\beta\dot{q}^\gamma + \frac{\partial g_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\gamma\dot{q}^\beta\right) - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\dot{q}^\beta\dot{q}^\gamma$$

$$= \frac{d}{dt}\left[\frac{1}{2}\left(g_{\alpha\beta}\dot{q}^\beta + g_{\alpha\gamma}\dot{q}^\gamma\right)\right] - \frac{\partial T}{\partial q^\alpha}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha}.$$

$$\therefore\ \boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = f_\alpha \quad (\alpha = 1, \cdots, N), \\ &T := \frac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta \quad \text{運動エネルギー}, \\ &f_\alpha := \sum_{i=1}^{N_p}\mathbb{F}_i \cdot \frac{\partial \mathbb{r}}{\partial q^\alpha} \quad (\alpha = 1, \cdots, N). \end{aligned}}$$

とくに 外力 が ポテンシャル $V(\mathbb{r}_1, \cdots, \mathbb{r}_{N_b})$ による場合

$$\mathbb{F}_i = -\frac{\partial V}{\partial \mathbb{r}_i} := -\left(\frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial y_i}, \frac{\partial V}{\partial z_i}\right).$$

$$\mathcal{F}_\alpha = -\sum_{i=1}^{N_b}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q^\alpha} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q^\alpha} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q^\alpha}\right)$$

$$= -\frac{\partial V}{\partial q^\alpha},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = -\frac{\partial V}{\partial q^\alpha}.$$

$\frac{\partial V}{\partial \dot{q}^\alpha} = 0$ より、

Lagrange運動方程式

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0 \quad (\alpha = 1, \cdots, N),$$

$L := T - V$ Lagrangian.

… 座標のとり方によらない形

2022 12 1 (木)

とくに 外力 が ポテンシャル $V(\mathbf{r}_1, \cdots, \mathbf{r}_{N_b})$ による場合

$$\mathbf{F}_i = -\frac{\partial V}{\partial \mathbf{r}_i} := -\left(\frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial y_i}, \frac{\partial V}{\partial z_i}\right).$$

$$F_\alpha = -\sum_{i=1}^{N_b}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q^\alpha} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q^\alpha} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q^\alpha}\right)$$

$$= -\frac{\partial V}{\partial q^\alpha},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = -\frac{\partial V}{\partial q^\alpha}.$$

$\frac{\partial V}{\partial \dot{q}^\alpha} = 0$ より,

**Lagrange 運動方程式**

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0 \quad (\alpha = 1, \cdots, N), \qquad L := T - V \quad \text{Lagrangian.}}$$

…座標のとり方によらない形

2022/12/2(金)

Christoffel 記号の変換則を証明する.

$$\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial q^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\alpha}\right)$$

の変換則を求める.

$$\tilde{g}_{\alpha\beta} = \frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta} g_{\kappa\lambda}.$$

$$\frac{\partial \tilde{g}_{\alpha\beta}}{\partial Q^\gamma} = \frac{\partial}{\partial Q^\gamma}\left(g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta}\right),$$

$$\frac{\partial \tilde{g}_{\alpha\beta}}{\partial Q^\gamma} = \frac{\partial g_{\kappa\lambda}}{\partial Q^\gamma}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta} + \underbrace{g_{\kappa\lambda}\frac{\partial^2 q^\kappa}{\partial Q^\gamma \partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta}}_{②} + g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}, \quad \text{—(1)}$$

$$\frac{\partial \tilde{g}_{\alpha\gamma}}{\partial Q^\beta} = \frac{\partial g_{\kappa\lambda}}{\partial Q^\beta}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\gamma} + \underbrace{g_{\kappa\lambda}\frac{\partial^2 q^\kappa}{\partial Q^\beta \partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\gamma}}_{①} + g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial^2 q^\lambda}{\partial Q^\gamma \partial Q^\beta} \quad \text{—(2)}$$

$$\frac{\partial \tilde{g}_{\beta\gamma}}{\partial Q^\alpha} = \frac{\partial g_{\kappa\lambda}}{\partial Q^\alpha}\frac{\partial q^\kappa}{\partial Q^\beta}\frac{\partial q^\lambda}{\partial Q^\gamma} + \underbrace{g_{\kappa\lambda}\frac{\partial^2 q^\kappa}{\partial Q^\alpha \partial Q^\beta}\frac{\partial q^\lambda}{\partial Q^\gamma}}_{①} + \underbrace{g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\beta}\frac{\partial^2 q^\lambda}{\partial Q^\gamma \partial Q^\alpha}}_{②}. \quad \text{—(3)}$$

$((1)+(2)-(3)) \times \frac{1}{2}$ をつくる,

$$\tilde{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\kappa\lambda}}{\partial Q^\gamma}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta} + \frac{\partial g_{\kappa\lambda}}{\partial Q^\beta}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\gamma} - \frac{\partial g_{\kappa\lambda}}{\partial Q^\alpha}\frac{\partial q^\kappa}{\partial Q^\beta}\frac{\partial q^\lambda}{\partial Q^\gamma}\right.$$

$$\left. + 2 g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{\kappa\lambda}}{\partial q^\mu}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma} + \frac{\partial g_{\kappa\lambda}}{\partial q^\mu}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\gamma}\frac{\partial q^\mu}{\partial Q^\beta}\right.$$

$$\left. - \frac{\partial g_{\kappa\lambda}}{\partial q^\mu}\frac{\partial q^\kappa}{\partial Q^\beta}\frac{\partial q^\lambda}{\partial Q^\gamma}\frac{\partial q^\mu}{\partial Q^\alpha}\right) + g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}$$

$$= \underbrace{\frac{1}{2}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma}\left(\frac{\partial g_{\kappa\lambda}}{\partial q^\mu} + \frac{\partial g_{\kappa\mu}}{\partial q^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial q^\kappa}\right)}_{\Gamma_{\kappa\lambda\mu}} + g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\alpha}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}$$

これより

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \tilde{g}^{\alpha\delta}\tilde{\Gamma}_{\delta\beta\gamma} = \frac{\partial Q^\alpha}{\partial q^\rho}\frac{\partial Q^\delta}{\partial q^\sigma} g^{\rho\sigma}\left(\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma}\Gamma_{\kappa\lambda\mu} + g_{\kappa\lambda}\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}\right)$$

$$= \frac{\partial Q^\alpha}{\partial q^\rho}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma} g^{\rho\kappa}\Gamma_{\kappa\lambda\mu} + \frac{\partial Q^\alpha}{\partial q^\rho}\underbrace{g^{\rho\kappa} g_{\kappa\lambda}}_{\delta^\rho_\lambda}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}$$

$$= \frac{\partial Q^\alpha}{\partial q^\rho}\frac{\partial q^\lambda}{\partial Q^\beta}\frac{\partial q^\mu}{\partial Q^\gamma}\Gamma^\rho_{\lambda\mu} + \frac{\partial Q^\alpha}{\partial q^\lambda}\frac{\partial^2 q^\lambda}{\partial Q^\beta \partial Q^\gamma}. \quad \boxtimes$$