力学からの微分幾何学(2)付録

共変微分が反変ベクトル場の成分の変換則に従うことの証明

ベクトル場

$$\mathbb{X} = X^\alpha \frac{\partial \boldsymbol{r}}{\partial q^\alpha} = \tilde{X}^\alpha \frac{\partial \boldsymbol{r}}{\partial Q^\alpha}, \quad \mathbb{Y} = Y^\alpha \frac{\partial \boldsymbol{r}}{\partial q^\alpha} = \tilde{Y}^\alpha \frac{\partial \boldsymbol{r}}{\partial Q^\alpha}$$

に対し

$$\nabla_X Y^\alpha = X^\beta \frac{\partial Y^\alpha}{\partial q^\beta} + \Gamma^\alpha_{\beta\gamma} X^\beta Y^\gamma,$$

$$\widetilde{\nabla_X Y^\alpha} = \tilde{X}^\beta \frac{\partial \tilde{Y}^\alpha}{\partial Q^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} \tilde{X}^\beta \tilde{Y}^\gamma$$

とおくと、

$$\widetilde{\nabla_X Y^\alpha} = \frac{\partial Q^\alpha}{\partial q^\nu} \nabla_X Y^\nu .$$

(証明) $\widetilde{\nabla_X Y^\alpha} = \tilde{X}^\beta \frac{\partial \tilde{Y}^\alpha}{\partial Q^\beta} + \tilde{\Gamma}^\alpha_{\beta\gamma} \tilde{X}^\beta \tilde{Y}^\gamma$

$$= \frac{\partial Q^\beta}{\partial q^\lambda} X^\lambda \frac{\partial}{\partial Q^\beta}\left(\frac{\partial Q^\alpha}{\partial q^\nu} Y^\nu\right) + \tilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} X^\lambda Y^\mu$$

$$= \frac{\partial Q^\beta}{\partial q^\lambda} X^\lambda \left(\frac{\partial Q^\alpha}{\partial q^\nu} \frac{\partial Y^\nu}{\partial Q^\beta} + \frac{\partial}{\partial Q^\beta}\left(\frac{\partial Q^\alpha}{\partial q^\nu}\right) Y^\nu\right) + \tilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} X^\lambda Y^\mu$$

$$= \frac{\partial Q^\alpha}{\partial q^\nu} X^\lambda \underbrace{\frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Y^\nu}{\partial Q^\beta}}_{\frac{\partial Y^\nu}{\partial q^\lambda}} + \underbrace{\frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial}{\partial Q^\beta}}_{\frac{\partial}{\partial q^\lambda}}\left(\frac{\partial Q^\alpha}{\partial q^\nu}\right) X^\lambda Y^\nu + \tilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} X^\lambda Y^\mu$$

$$= \frac{\partial Q^\alpha}{\partial q^\nu} X^\lambda \frac{\partial Y^\nu}{\partial q^\lambda} + \underbrace{\frac{\partial^2 Q^\alpha}{\partial q^\nu \partial q^\lambda} X^\lambda Y^\nu}_{①} + \underbrace{\tilde{\Gamma}^\alpha_{\beta\gamma} \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} X^\lambda Y^\mu}_{②} .$$

$$① = \delta^\alpha_\beta \frac{\partial^2 Q^\beta}{\partial q^\nu \partial q^\lambda} X^\lambda Y^\nu = \frac{\partial Q^\alpha}{\partial q^\mu} \frac{\partial q^\mu}{\partial Q^\beta} \frac{\partial^2 Q^\beta}{\partial q^\nu \partial q^\lambda} X^\lambda Y^\nu$$

$$= \frac{\partial Q^\alpha}{\partial q^\nu} \frac{\partial q^\nu}{\partial Q^\beta} \frac{\partial^2 Q^\beta}{\partial q^\lambda \partial q^\mu} X^\lambda Y^\mu ,$$

$$② = \delta^\alpha_\delta \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} \tilde{\Gamma}^\delta_{\beta\gamma} X^\lambda Y^\mu$$

$$= \frac{\partial Q^\alpha}{\partial q^\nu} \frac{\partial q^\nu}{\partial Q^\delta} \frac{\partial Q^\beta}{\partial q^\lambda} \frac{\partial Q^\gamma}{\partial q^\mu} \tilde{\Gamma}^\delta_{\beta\gamma} X^\lambda Y^\mu$$

により、

$$\widetilde{\nabla_X Y^\alpha} = \frac{\partial Q^\alpha}{\partial q^\kappa}\left[X^\lambda \frac{\partial Y^\kappa}{\partial q^\lambda} + \underbrace{\left(\frac{\partial q^\kappa}{\partial Q^\delta}\frac{\partial Q^\beta}{\partial q^\lambda}\frac{\partial Q^\gamma}{\partial q^\mu}\tilde{\Gamma}^\delta_{\beta\gamma} + \frac{\partial q^\kappa}{\partial Q^\beta}\frac{\partial^2 Q^\beta}{\partial q^\lambda \partial q^\mu}\right)}_{Ⓐ} X^\lambda Y^\mu\right].$$

Christoffelの記号の変換則（第1回動画「力学からの微分幾何学」付録参照）より Ⓐ $= \Gamma^\kappa_{\lambda\mu}$ であるから.

$$\widetilde{\nabla_X Y^\alpha} = \frac{\partial Q^\alpha}{\partial q^\kappa}\left(X^\lambda \frac{\partial Y^\kappa}{\partial q^\lambda} + \Gamma^\kappa_{\lambda\mu} X^\lambda Y^\mu\right) = \frac{\partial Q^\alpha}{\partial q^\kappa} \nabla_X Y^\kappa. \qquad \boxtimes$$